37. *Discrete Mean Values of Hurwitz Zeta-functions*

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1. The results. Let $\zeta(s, \alpha)$ be the Hurwitz zeta-function with a positive parameter α, and $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$. The behaviour of the integral

$$I(t) = \int_0^1 \left| \zeta_1\left(\frac{1}{2} + it, \alpha\right) \right|^2 d\alpha$$

has been studied by various authors. Zhang [5] [8] conjectured that for any $t \geq 1$,

$$I(t) = \log(t/2\pi) + \gamma + O(t^{-1/4}), \tag{1.1}$$

where γ is Euler's constant. (Perhaps this conjecture had been well-known among Indian number theorists.) Quite recently, Zhang [11] proved this conjecture; indeed, he has shown the following far better result:

$$I(t) = \log(t/2\pi) + \gamma - 2\mathrm{Re}\frac{\zeta\left(\frac{1}{2} + it\right)}{\frac{1}{2} + it} + O(t^{-1}), \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta-function.

Let q be a positive integer. In this note we consider the discrete mean value

$$J(s, q) = \sum_{1 \leq a \leq q} | \zeta(s, a/q) |^2.$$

Let $\Gamma(s)$ be the gamma-function, $\psi(s) = (\Gamma'/\Gamma)(s)$, N be a positive integer, and define

$$R_N(u, v; q) = \frac{1}{\Gamma(u)\Gamma(v)} \int_0^\infty \frac{y^{v+N-1}}{e^y - 1} \times$$
$$\times \int_0^\infty \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x + q^{-1}\tau y) x^{u-1}\, d\tau\, dx dy$$

for $0 < \mathrm{Re}\, u < N + 1$ and $\mathrm{Re}\, v > -N + 1$, where $h^{(N)}(z)$ is the N-th derivative of

$$h(z) = \frac{e^z}{e^z - 1} - \frac{1}{z}.$$

Then we have

Theorem 1. *For any $t \geq 1$ and any positive integers N and q, we have*

(1.3) $J\left(\frac{1}{2} + it, q\right)$

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$$= q\left\{\log(q/2\pi) + 2\gamma + \mathrm{Re}\,\psi\left(\frac{1}{2} + it\right)\right\}$$

$$+ 2\sum_{n=0}^{N-1} \frac{(-1)^n q^{-n}}{n!} \mathrm{Re}\left\{q^{\frac{1}{2}+it} \frac{\Gamma\left(\frac{1}{2} - it + n\right)}{\Gamma\left(\frac{1}{2} - it\right)} \zeta\left(\frac{1}{2} + it - n\right)\zeta\left(\frac{1}{2} - it + n\right)\right\}$$

$$+ 2q^{-N}\,\mathrm{Re}\left\{q^{\frac{1}{2}+it} R_N\left(\frac{1}{2} + it, \frac{1}{2} - it\,;\, q\right)\right\}.$$

Remark. Since $\mathrm{Re}\,\psi\left(\frac{1}{2} + it\right) = \log t + O(t^{-2})$, the first term in the right-hand side can be written as $q\{\log(qt/2\pi) + 2\gamma + O(t^{-2})\}$. Also, Katsurada's result [1] implies

$$R_N(\sigma + it, \sigma - it\,;\, q) = O(t^{2N+\frac{1}{2}-\sigma}) \quad (0 < \sigma < 1)$$

with the O-constant depending only on σ and N. Therefore, the above theorem gives the asymptotic expansion of $J\left(\frac{1}{2} + it,\, q\right)$ with respect to q.

Zhang also studied $J\left(\frac{1}{2} + it,\, q\right)$ in his papers [4] [6] [7] [9] [10]. For example, in [9] he proved

$$J\left(\frac{1}{2} + it,\, q\right) = q\{\log(qt/2\pi) + 2\gamma\} + O(qt^{-\frac{1}{12}}) + O((t^{\frac{5}{6}} + q^{\frac{1}{2}}t^{\frac{5}{12}})\log^3 t),$$

which should be compared with our Theorem 1.

In the next section we will prove

Theorem 2. *For any $t \geq 1$, any positive integers N, q and any σ satisfying $0 < \sigma < 1$, $\sigma \neq \frac{1}{2}$, we have*

(1.4) $J(\sigma + it,\, q)$

$$= q^{2\sigma}\zeta(2\sigma) + 2q\Gamma(2\sigma - 1)\zeta(2\sigma - 1)\,\mathrm{Re}\left\{\frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)}\right\}$$

$$+ 2\sum_{n=0}^{N-1} \frac{(-1)^n q^{-n}}{n!} \mathrm{Re}\left\{q^{\sigma+it} \frac{\Gamma(\sigma - it + n)}{\Gamma(\sigma - it)} \zeta(\sigma + it - n)\zeta(\sigma - it + n)\right\}$$

$$+ 2q^{-N}\,\mathrm{Re}\,\{q^{\sigma+it} R_N(\sigma + it, \sigma - it\,;\, q)\}.$$

Theorem 1 can be easily deduced from Theorem 2 as the limit case $\sigma \to \frac{1}{2}$. Our proof is an analogue of the argument in [2], so it is a variant of Atkinson's method. The basic idea of this variant is due to Motohashi [3]. It seems that Zhang's method [11] is not easily applicable to the discrete mean $J(s,\, q)$. It should be noted that our method can be modified so as to obtain an alternative proof of (1.1) and (1.2). (See the last section.)

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2. Proof of Theorem 2. The argument is quite similar as that in [2], so we give only a brief sketch. Let u, v be complex variables, and

$$B(u, v ; q) = \sum_{1\le a\le q} \zeta(u, a/q)\zeta(v, a/q).$$

First we assume $\mathrm{Re}\, u > 1$, $\mathrm{Re}\, v > 1$. Then we can divide

(2.1) $$B(u, v ; q) = q^{u+v}\zeta(u+v) + \varphi(u, v ; q) + \varphi(v, u ; q),$$

where

$$\varphi(u, v ; q) = q^{u+v} \sum_{1\le a\le q} \sum_{m=0}^{\infty} (qm+a)^{-u} \sum_{n=1}^{\infty} (q(m+n)+a)^{-v}.$$

The function $\varphi(u, v ; q)$ has the analytic continuation of the form

$$\varphi(u, v ; q) = \frac{1}{\Gamma(u)\Gamma(v)} \sum_{1\le a\le q} \int_0^\infty \frac{y^{v-1}}{e^y-1} \int_0^\infty \frac{e^{(1-(a/q))(x+y)}}{e^{x+y}-1} x^{u-1}\, dxdy$$

in the region $\mathrm{Re}\, u > 0$, $\mathrm{Re}\, v > 1$, $\mathrm{Re}(u+v) > 2$, which can be proved by using the fact

$$\int_0^\infty \frac{e^{(1-(a/q))(x+y)}}{e^{x+y}-1} x^{u-1}\, dx = \Gamma(u) \sum_{m=0}^{\infty} e^{-(\frac{a}{q}+m)y}\left(m+\frac{a}{q}\right)^{-u} \quad (\mathrm{Re}\, u > 0,\ y > 0).$$

Hence, putting

$$h(z ; \alpha) = \frac{e^{(1-\alpha)z}}{e^z-1} - \frac{1}{z},$$

we obtain

(2.2)
$$\varphi(u, v ; q) = \sum_{1\le a\le q} \left\{\Gamma(u+v-1)\zeta(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)} + g\left(u, v ; \frac{a}{q}\right)\right\},$$

where

$$g(u, v ; \alpha) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)} \int_{\mathscr{C}} \frac{y^{v-1}}{e^y-1} \times$$
$$\times \int_{\mathscr{C}} h(x+y ; \alpha) x^{u-1}\, dxdy$$

and the contour $\mathscr{C}$ is the same as in [2]. For any $\alpha > 0$, the above integral converges absolutely for $\mathrm{Re}\, u < 1$ and any v, so (2.2) is valid in this region. Using the formula

$$\zeta(s, \alpha) = \frac{1}{\Gamma(s)(e^{2\pi is}-1)} \int_{\mathscr{C}} \frac{y^{s-1}e^{(1-\alpha)y}}{e^y-1}\, dy \quad (\alpha > 0),$$

we have

$$\int_{\mathscr{C}} h(x ; \alpha) x^{u-1} dx = (e^{2\pi iu}-1)\Gamma(u)\zeta(u, \alpha)$$

for $\mathrm{Re}\, u < 1$, hence

(2.3)
$$g(u, v ; \alpha) = \zeta(u, \alpha)\zeta(v) + \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)} \int_{\mathscr{C}} \frac{y^{v-1}}{e^y-1} \times$$
$$\times \int_{\mathscr{C}} (h(x+y ; \alpha) - h(x ; \alpha)) x^{u-1}\, dxdy.$$

Therefore, noting

$$\sum_{1\le a\le q} h(z ; a/q) = h(z/q) - 1$$

we find, as an analogue of [2, (2.7)],

$$\begin{aligned}\varphi(u, v; q) &= q\Gamma(u+v-1)\zeta(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)} + q^u\zeta(u)\zeta(v) \\ &+ \frac{q^u}{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)}\int_{\mathscr{C}}\frac{y^{v-1}}{e^y-1}\int_{\mathscr{C}}(h(x+q^{-1}y) - \\ &\qquad h(x))x^{u-1}\,dxdy\end{aligned}$$

for $\mathrm{Re}\, u < 1$ and any v. The last term can be handled in the same way as G_2 in Section 4 of [2]. Substituting the results into (2.1), we obtain

$$\begin{aligned}(2.4)\quad & B(u, v; q) \\ &= q^{u+v}\zeta(u+v) + q\Gamma(u+v-1)\zeta(u+v-1)\times \\ &\qquad \times\left(\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right) \\ &+ \sum_{n=0}^{N-1}\frac{(-1)^n q^{-n}}{n!}\left\{q^u\frac{\Gamma(v+n)}{\Gamma(v)}\zeta(u-n)\zeta(v+n)\right. \\ &+ \left. q^v\frac{\Gamma(u+n)}{\Gamma(u)}\zeta(v-n)\zeta(u+n)\right\} \\ &+ q^{-N}\{q^u R_N(u, v; q) + q^v R_N(v, u; q)\}\end{aligned}$$

for $\mathrm{Re}\, u < 1$, $\mathrm{Re}\, v < 1$ and any positive integer N. Theorem 2 is just the case $u = \sigma + it$ and $v = \sigma - it$.

3. Remarks. The formula [2,(4.4)] implies that $R_N(u, v; q)$ can be continued meromorphically to $\mathrm{Re}\, u < N+1$ and any v. Hence, (2.4) is valid for $\mathrm{Re}\, u < N+1$ and $\mathrm{Re}\, v < N+1$. Therefore, (1.4) is actually valid for any $\sigma < N+1$ and any real t, except for the points at which some singularity appears in the right-hand side. The singular cases can be treated, as Theorem 1, as the limit cases.

We can also treat the sum

$$\sum_{\substack{1\le a\le q\\ (a,q)=1}} |\zeta(s, a/q)|^2$$

by the same method. In fact, the asymptotic formula of this sum can be deduced as a direct consequence of the results in [2], because the above sum is equal to

$$\varphi(q)^{-1}q^{2\sigma}\sum_{\chi(\mathrm{mod}\, q)}|L(s,\chi)|^2,$$

where $L(s, \chi)$ is the Dirichlet L-function with a character χ mod q, and $\varphi(q)$ is Euler's function.

Finally, we return to the problem of evaluating the continuous mean value $I(t)$. As an analogue of [2,(2.2)](which is originally due to Motohashi [3]), we can show

$$\begin{aligned}(3.1)\quad & \zeta_1(u, \alpha)\zeta_1(v, \alpha) \\ &= \zeta(u+v, \alpha+1) + \Gamma(u+v-1)\zeta(u+v-1)\times \\ &\qquad \times\left(\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right) \\ &\quad + g(u, v; \alpha) + g(v, u; \alpha) - \alpha^{-u}\zeta(v, \alpha+1) - \alpha^{-v}\zeta(u, \alpha+1)\end{aligned}$$

for $\mathrm{Re}\, u < 1$, $\mathrm{Re}\, v < 1$. From (2.3) we have

$$g(u, v; \alpha) = \zeta(u, \alpha)\zeta(v) + \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \int_{\mathscr{C}} \frac{y^v}{e^y - 1} \times$$
$$\times \int_{\mathscr{C}} \int_0^1 h'(x + \tau y; \alpha) x^{u-1} \, d\tau \, dx dy.$$

The estimate $h'(x + \tau y, \alpha) = O(\alpha e^{-\alpha|x|} + (1 + |x|)^{-2})$ holds for any $x, y \in \mathscr{C}$, $0 \leq \tau \leq 1$ and $0 \leq \alpha \leq 1$. Therefore, by using the facts

$$\int_0^1 h'(x + \tau y, \alpha) d\alpha = 0$$

and

$$\int_0^1 \zeta(s, \alpha) d\alpha = 0 \quad (\mathrm{Re}\, s < 1)$$

it follows that

$$\int_0^1 g(u, v; \alpha) d\alpha = 0.$$

Hence from (3.1) we have

$$(3.2) \quad \int_0^1 |\zeta_1(\sigma + it, \alpha)|^2 d\alpha$$
$$= \int_0^1 \zeta(2\sigma, \alpha + 1) d\alpha$$
$$+ \Gamma(2\sigma - 1)\zeta(2\sigma - 1)\left(\frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)} + \frac{\Gamma(1 - \sigma - it)}{\Gamma(\sigma - it)}\right)$$
$$- \int_0^1 \alpha^{-\sigma - it}\zeta(\sigma - it, \alpha + 1) d\alpha - \int_0^1 \alpha^{-\sigma + it}\zeta(\sigma + it, \alpha + 1) d\alpha$$

for $0 < \sigma < 1$, which is essentially the same as Lemma 2 in Zhang [11]. Applying the approximate functional equation of $\zeta_1(s, \alpha)$ to the last two integrals, we can show that those integrals are $O(t^{-\frac{1}{4}})$. Thus the authors proved the conjecture (1.1), independently of Zhang, in the last December. But the authors overlooked the simple argument of Lemma 1 in Zhang [11]. Applying Zhang's lemma to the above integrals, we can give an alternative proof of (1.2). Further results in this direction will appear in forthcoming papers.

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