3. Generalization of a Theorem of Manin-Shafarevich*)

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Let us fix some notation before stating the results. Let m be a positive integer with $m \geq 3$. Let $\{\Gamma_t \mid t \in \boldsymbol{P}^1\}$ be a linear pencil of curves of degree m in a projective plane $\boldsymbol{P}^2$ defined over an algebraically closed field k of arbitary characteristic. Assume the following conditions:

(A1) Every member Γ_t is irreducible and general members are nonsingular.

(A2) The m^2 base points of the pencil are distinct. We denote them by $P_i (i = 0,1,\ldots,m^2 - 1)$.

Then the generic member $\Gamma = \Gamma_t$ (for t generic over k) is a nonsingular curve of genus $g = (m-1)(m-2)/2$ defined over the rational function field $K = k(t)$.

Let J denote the Jacobian variety of Γ/K and $J(K)$ the group of its K-rational points. Each P_i defines a K-rational point of Γ. By choosing one of P_i, say P_0, we have a natural embedding of Γ into J sending P_0 to the origin of J. Thus we have

$$P_1, \ldots, P_{m^2-1} \in \Gamma(K) \subset J(K).$$

For $m = 3$, $\{\Gamma_t\}$ is a pencil of cubic curves and $J = \Gamma$ is an elliptic curve, say E, over K. Inspired by Shafarevich, Manin proved that under (A1) and (A2) the 8 points $P_1, \ldots, P_8$ are independent and generate a subgroup of index 3 in the Mordell-Weil group $E(K)$ (see [5], Th.6 and [6], Ch.IV, 26.4). Recently we have given a simple proof of this result based on the theory of Mordell-Weil lattices, where $E(K)$ is endowed with the structure of the root lattice E_8 (see [7], Th. 10.11).

More recently we have extended the notion of Mordell-Weil lattices to higher genus case [9]. As an application, we can prove the following result generalizing the above theorem of Manin-Shafarevich to arbitary $m \geq 3$.

Theorem 1. *The notation being as above, assume the conditions* (A1) *and* (A2). *Then the group of K-rational points $J(K)$ of the Jacobian variety J is a torsionfree abelian group of rank $r = m^2 - 1$, and the r points $P_i (1 \leq i \leq r)$ are independent and generate a subgroup of index m in $J(K)$.*

This is an immediate consequence of Theorem 2 below formulated in terms of Mordell-Weil lattices. By blowing up the m^2 base points from $\boldsymbol{P}^2$, we obtain a nonsingular rational surface S and a morphism

$$f : S \to \boldsymbol{P}^1$$

such that $f^{-1}(t) \simeq \Gamma_t (t \in \boldsymbol{P}^1)$. In particular, Γ/K is the generic fibre of this genus g fibration f. The exceptional curves (P_i) in S arising from

*) Dedicated to I. R. Shafarevich for his 70th birthday.

$P_i \in \boldsymbol{P}^2$ define m^2 sections of f. We choose $(P_0) = (O)$ as the zero-section.

Theorem 2. *With respect to the height pairing defined in* [9], *the Mordell-Weil lattice* $L = J(K)$ *is a positive-definite integral unimodular lattice of rank* $r = m^2 - 1$. *It is an even lattice if and only if* m *is odd. The* r *points* $P_i \in L$ *generate a sublattice of index* m *in* L. *There is a unique point* $Q \in L$ *such that* $mQ = P_1 + \cdots + P_r$, *and* $P_1, \ldots, P_{r-1}$, Q *form a set of free generators of* $L = J(K)$.

Proof. First we note that the K/k-trace of the Jacobian J/K is trivial (i.e. the condition (∗) of [9] is satisfied). Indeed this is the case for any fibration $f : S \to \boldsymbol{P}^1$ where S is a rational surface. This fact will be explained in the detailed version of [9]. Hence $J(K)$ is finitely generated (Mordell-Weil theorem for function fields; see [4], Ch.6).

Now the condition (A1) implies that f has no reducible fibres. Hence $J(K)$ coincides with the narrow Mordell-Weil lattice $J(K)^0$, which is always a (torsionfree) positive-definite integral lattice. Moreover $L = J(K)$ is a unimodular lattice of rank $r = m^2 - 1$ since the Néron-Severi lattice of S is unimodular and of rank $\rho(S) = 1 + m^2$.

The rest of the proof is parallel to that given for the case $m = 3$ in [7], Th.10.11. First the height pairing of the points P_i can be computed as follows. By the formula (9) in [9], we have

$$\langle P_i, P_j \rangle = -(O^2) - (P_i P_j) + (P_i O) + (P_j O) = \begin{cases} 2 & i = j\ (\geq 1) \\ 1 & i \neq j \end{cases}$$

using the obvious fact that $(P_i^2) = -1$ and $(P_i) \cap (P_j) = \emptyset$ for $i \neq j$. Then it is easy to compute the Gram determinant

$$\det(\langle P_i, P_j \rangle) = m^2 \neq 0.$$

This shows that $P_1, \ldots, P_r$ are independent and that they span a sublattice, say H, of index m in the unimodular lattice L.

Next take any point $Q \in L - H$. Then $mQ \in H$ can be written as $\sum_i n_i P_i$ for some integers n_i. Since L is an integral lattice, $\langle Q, P_i \rangle$ is an integer for any i. This implies that $n_i \equiv n_j \bmod m$ for any i, j. Hence we have $mQ = \nu(P_1 + \cdots + P_r) + mR$ for some $\nu \in \boldsymbol{Z}$ and some $R \in H$.

Thus there is a point $Q \in L$ such that $mQ = P_1 + \cdots + P_r$, which is unique since L is torsionfree. It is clear that L is generated by $P_1, \ldots, P_{r-1}$ and Q. Hence L is an even lattice if and only if $\langle Q, Q \rangle$ is even. But we have

$$\langle Q, Q \rangle = m^2 - 1,$$

since the norm of $P_1 + \cdots + P_r$ is equal to $r(r+1) = m^2(m^2 - 1)$. Therefore L is an even lattice if and only if m is odd. This completes the proof of Theorem 2.

Remark. (i) Note that Theorem 1 or 2 is not vacuous because for any $m \geq 3$ there exist linear pencils of plane curves of degree m satisfying the conditions (A1) and (A2). This can be verified by an elementary dimension-count argument (for this we had a useful discussion with K. Oguiso). More generally, existence of such a pencil follows from theory of Lefschetz pencils (see e. g. [1] or [3]).

(ii) In the classical case $m = 3$, it is easy to find pencils defined over the rational number field $\boldsymbol{Q}$ such that the 9 base points are $\boldsymbol{Q}$-rational, giving rise to an elliptic curve over $\boldsymbol{Q}(t)$ of rank 8. This fact was used in our effective version of Néron's method for constructing elliptic curves with high rank ([8]).

Question. For $m \geq 4$, does there exist a pencil of degree m curves, defined over $\boldsymbol{Q}$, satisfying (A1), (A2) such that all the m^2 base points are $\boldsymbol{Q}$-rational?

Actually the above proof works in a more general context. Namely, combined with the idea of Lefschetz pencils as in Remark (i), we can prove the following result which will be proved in detail elsewhere.

Theorem 3. *Let X be a smooth algebraic surface with a trivial Picard variety, embedded as a surface of degree d in a projective space $\boldsymbol{P}^N$. Suppose that $\{\Gamma_t \mid t \in \boldsymbol{P}^1\}$ is a Lefschetz pencil of hyperplane sections of X. Let J denote the Jacobian variety of the generic member of this pencil, say Γ, defined over the rational function field $K = k(t)$. Then $J(K)$ is a positive-definite integral lattice of rank*

$$r = \rho(X) + d - 2$$

whose determinant is equal to $|\det \mathrm{NS}(X)|$.

The previous result corresponds to the case where X is the isomorphic image of $\boldsymbol{P}^2$ under the embedding defined by the complete linear system $|mH|$ (H: a line in $\boldsymbol{P}^2$) so that $d = m^2$.

Example. Let X be the Fermat surface of degree 4 in $\boldsymbol{P}^3$ in characteristic 0. We know that X is a $K3$ surface with $\rho(X) = 20$, $|\det \mathrm{NS}(X)| = 64$ (cf. [2]). Then Γ is a curve of genus 3 and its Jacobian variety has the Mordell-Weil group $J(K)$ of rank 22 with $\det = 64$.

References

[1] Igusa, J.: Fibre systems of Jacobian varieties. Am. J. Math., **78**, 171–199 (1956).

[2] Inose, H., and Shioda, T.: On singular K3 surfaces. Complex Analysis and Algebraic Geometry. Iwanami and Cambridge Univ. Press, pp. 119–136 (1977).

[3] Katz, N.: Pinceaux de Lefschetz: théorème d'existence. SGA 7 II, Lect. Notes in Math., vol. 340, pp. 212–253 (1973).

[4] Lang, S.: Fundamentals of Diophantine Geomentry. Springer-Verlag (1983).

[5] Manin, Ju.: The Tate height of points on an Abelian variety, its variants and applications. Izv. Akad. Nauk SSSR, ser. Mat., **28**, 1363–1390 (1964); A. M. S. Transl., (2) **59**, 82–110 (1966).

[6] ——: Cubic Forms. North-Holland (1974).

[7] Shioda, T.: On the Mordell-Weil lattices. Comment. Math. Univ. St. Pauli, **39**, 211–240 (1990).

[8] ——: An infinite family of elliptic curves over $\boldsymbol{Q}$ with large rank via Neron's method. Invent. Math., **106**, 109–119 (1991).

[9] ——: Mordell-Weil lattices for higher genus fibration. Proc. Japan Acad., **68**A, 247–250 (1992).