1. Complete Kähler Manifolds with Zero Ricci Curvature and Kobayashi-Ochiai's Characterization of Complex Projective Spaces

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In [3] (cf. [9]), Bando and the author proved that there exists a complete Ricci-flat Kähler metric on the complement of a smooth hypersurface D of a Fano manifold X if (X, D) satisfies the conditions: (i) $c_1(X) = \alpha[D]$ with $\alpha > 1$ and (ii) D admits a Kähler-Einstein metric. This result and its proof find some applications in [2], [5], [6] and [11] to problems differential geometry. But we find this existence theorem quite restrictive if we try to apply it to problems in complex algebraic geometry. In this note we announce a general existence theorem for complete Ricci-flat Kähler metrics on certain class of affine algebraic manifolds, which generalizes the results in [3] and [9] by removing the Kähler-Einstein condition at infinity. Details and an application will appear elsewhere ([7]).

Theorem 1. *Let X be a Fano manifold, i.e., X has ample anticanonical bundle. Let D be a smooth connected hypersurface in X such that $c_1(X) = \alpha[D]$ with $\alpha > 1$. Then there exists a complete Ricci-flat Kähler metric on $X - D$.*

The asymptotic bahavior of the resulting Ricci-flat Kähler metric may be described as follows. As $c_1(X) > 0$ and $c_1(X) = \alpha[D]$, there exists a Hermitian metric $\|\cdot\|$ on the line bundle $O_X(D)$ such that

$$\theta = \sqrt{-1}\partial\bar{\partial}t \left(t = \log \frac{1}{\|\sigma\|^2}\right)$$

defines a Kähler metric on X, where σ is a holomorphic section of $O_X(D)$ vanishing along D. Then

$$\begin{aligned}\omega &= \sqrt{-1}\partial\bar{\partial}\frac{n}{\alpha-1}\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}}\\ &= \left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}}\left(\theta + \frac{\alpha-1}{n}\sqrt{-1}\partial t\wedge\bar{\partial}t\right)\end{aligned}$$

turns out to be a complete Kähler metric on $X - D$. The resulting Ricci-flat Kähler metric on $X - D$ has a Kähler potential $\tilde{u}$ of the form

$$\tilde{u} = \frac{n}{\alpha-1}\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}} + u$$

where u satisfies the *a priori* growth (decay, if $k \geq 3$) estimates:

$$\|\nabla_\omega^k u\| \leq C_k\left\{\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{2n}}\right\}^{2-k}$$

for $0 \leq \forall\, k \in Z$, where ∇_ω denotes the Levi-Civita connection of ω. In particular, u is at most of quadratic growth relative to the distance function of ω from a fixed point in $X-D$ and the complete Ricci-flat Kähler metric $\tilde{\omega}=\sqrt{-1}\partial\bar{\partial}\tilde{u}$ is equivalent to ω:

$$C\omega<\tilde{\omega}<C^{-1}\omega$$

holds with some *a priori* constant $C>0$. We expect that Theorem 1 will be useful in complex algebraic geometry. In fact, as we explain below, we are able to understand Kobayashi-Ochiai's characterization [8] of complex projective spaces from the view point of complete Ricci-flat Kähler metrics. Kobayashi-Ochiai's theorem is essentially as follows:

Theorem 2 ([8, p.32]). *Let X be an n-dimensional compact complex manifold with an ample line bundle L. Suppose*

$$c_1(X)\geq g\,c_1(L)$$

with $Z\ni g\geqslant n+1$. Then (X, L) is biholomorphic to the hyperplane section $(P_n(C), O(1))$.

Using the Hirzebruch-Riemann-Roch theorem and the Kodaira vanishing theorem, Kobayashi and Ochiai first showed

Lemma 1 ([8, p.36]). *Let (X, L) be as in Theorem 2. Then the following two properties hold:*

(1) $c_1(L)^n[X]=1$,

(2) $\dim H^0(X,L)=n+1$.

Then they showed (by elementary induction) that the above properties imply the following Lamma 2.

Lemma 2 ([8, Lemmma 2(I) and Lemma 3]). *Let X and L be as in Theorem 2 and let $\sigma_0,\cdots,\sigma_n$ be linearly independent elements of $H^0(X, L)$ with $D_0,\cdots,D_n$ their zero divisors. Then $V_{n-k-1}=D_0\cap D_1\cap\cdots\cap D_k (k=0,1,\cdots,k)$ is irreducible of dimension $n-k-1$ whose Poincaré dual is $(c_1(L))^{k+1}$. In particular $H^0(X, F)$ has no base points.*

Bertini's theorem then implies

Lemma 3. *The generic element of the linear system $|L|$ is a smooth irreducible hypersurface in X.*

Now we arrive at the following situation: X is a Fano manifold, there exists a smooth connected hypersurface D such that $c_1(X)=\alpha[D]$ with $\alpha\geq n+1$. From Theorem 1, we have

Lemma 4. *There exists a complete Ricci-flat Kähler metric $\tilde{\omega}$ on $X-D$ with asymptotic properties described above.*

Let D be defined by $\sigma_0=0$ and set $z_i=\frac{\sigma_1}{\sigma_0}$ for $1\leq i\leq n$. Then $\{z_i\}_{i=1}^n$ are nonconstant holomorphic functions on $X-D$ with at most linear growth with respect to the distance function of the metric $\tilde{\omega}$. Lemma 2 implies that $\eta=dz_1\wedge\cdots\wedge dz_n$ is a nonvanishing holomorphic n-form on $X-D$ with poles of order $n+1$ along D. Therefore the function

$$f=\log\frac{\eta\wedge\bar{\eta}}{\tilde{\omega}^n}$$

is a bounded pluriharmonic function on $X-D$ which extends smoothly on X. Thus f turns out to be a constant function. Hence each z_i is just of linear growth and $\alpha=n+1$. Now we consider the finite holomorphic map

$$z=(z_1,\cdots,z_n):X-D\to C^n.$$

As z is of maximal rank everywhere and C^n is simply connected, the map z must be an isomorphism. Moreover the Ricci-flat Kähler potential $\tilde{u}$ on $X-D$ is equivalent to $\| z \|^2 = | z_1 |^2 + \cdots + | z_n |^2$. Hence we have a Kähler potential $\tilde{u}$ on C^n which is equivalent to the squared distance function from the origin of the standard flat metric and satisfies the complex Monge-Ampère equation

$$\det\left(\frac{\partial^2 \tilde{u}}{\partial z_i \partial \bar{z}_j}\right) = 1.$$

Write $\tilde{u} = \| z \|^2 + u$ and $\omega = \sqrt{-1}\,\partial\bar{\partial} \| z \|^2$ and think of $\tilde{\omega}$ as a deformation of ω. Then Theorem 1 implies that $\tilde{\omega}$ is equivalent to ω. Set

$$\| \phi \|^2 = \tilde{g}^{\alpha\bar{\beta}} \tilde{g}^{\lambda\bar{\mu}} \tilde{g}^{\nu\bar{\gamma}} \nabla_\alpha \nabla_{\nu\bar{\mu}} u \nabla_{\bar{\beta}} \nabla_{\lambda\bar{\gamma}} u,$$

where ∇ denotes the Levi-Civita connection of the flat metric ω. Then Calabi-Aubin-Yau's identity ([4], [10] and [1, Lemma, p.153]) implies that $\| \phi \|^2$ is a nonnegative subharmonic function:

$$\Delta_\omega \| \phi \|^2 \geq 0.$$

But it follows from Theorem 1 that the third derivatives of $\tilde{u}$ are of order $\| z \|^{2-3}$. Hence the maximum principle implies that $\| \phi \|$ vanishes identically, i.e., the third derivatives of mixed type all vanish. As every function in $\mathrm{Ker}(\partial\bar{\partial})$ can be written as $f + \bar{g}$ with holomorphic functions f and g, we infer that $\tilde{u}$ is of the form

$$\tilde{u} = \textstyle\sum_{\alpha,\beta=1}^{n} \tilde{g}_{\alpha\bar{\beta}} z_\alpha \bar{z}_\beta + f + \bar{f}$$

where the coefficients $\tilde{g}_{\alpha\bar{\beta}}$ are constant and f is holomorphic. Therefore $\tilde{\omega}$ is a complete flat metric. So D has a tubular neighborhood in X diffeomorphic to S^{2n-1} and the J-rotation of the gradient vector field of the distance function (relative to $\tilde{\omega}$) is isotopic to the vertical vector field of the Hopf fibration. Thus X turns out to be diffeomorphic to $P_n(C)$. This implies that the holomorphic map $z: X - D \to C^n$ extends to a holomorphic map

$$[\sigma_0 : \sigma_1 : \cdots : \sigma_n] : X \to P_n(C)$$

which is a diffeomorphism. Thus (X, L) is biholomorphic to the hyperplane section $(P_n(C), O(1))$.

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