83. A Note on Certain Infinite Products

By Masao TOYOIZUMI

Department of Mathematics, Toyo University

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1. Statement of result. Let M be a positive integer, χ a real non-principal primitive character modulo M, $L(s,\chi)$ the associated L-series and $\zeta_M = \exp(2\pi i/M)$. Given a sequence $a(1), a(2), a(3), \cdots$ of integers such that $a(n) = O(n^c)$ for some $c > 0$, we define, for $Im(z) > 0$,

$$f_\chi(z) = \exp(2\pi iaz) \prod_{h=0}^{M-1} \prod_{n=1}^{\infty} (1 - \zeta_M^h q(\lambda)^n)^{\chi(h)a(n)}, \tag{1}$$

where $q(\lambda) = \exp(2\pi iz/\lambda)$, $\lambda > 0$ and a is a real number. Then the infinite product converges absolutely and uniformly in every compact subset of the upper half plane H. Hence $f_\chi(z)$ is holomorphic in H. To state our theorem, let $\phi(s)$ be a convergent Dirichlet series defined by

$$\phi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

Theorem. *Assume that $\phi(s)$ can be continued through the whole s-plane as a non-zero meromorphic function with a finite number of poles and that there exists a real number k such that*

$$f_\chi(-1/z) = (z/i)^k f_\chi(z). \tag{2}$$

Then $(\lambda/M)^2$ is an integer, $a = k = 0$ and $f_\chi(z)$ is given by

$$f_\chi(z) = \prod_{m\mid(\lambda/M)^2} \varphi_\chi(mz)^{c(m)}, \tag{3}$$

where

$$\varphi_\chi(z) = \prod_{h=0}^{M-1} \prod_{n=1}^{\infty} (1 - \zeta_M^h q(\lambda)^n)^{\chi(h)\chi(n)},$$

and $c(m)$, defined for m dividing $(\lambda/M)^2$, are integers such that $c(m) = \chi(-1)c((\lambda/M)^2/m)$ for any divisor m of $(\lambda/M)^2$.

Conversely, let $(\lambda/M)^2$ be an integer and let $c(m)$, for integers m dividing $(\lambda/M)^2$, be arbitrary integers such that $c(m) = \chi(-1)c((\lambda/M)^2/m)$ for any divisor m of $(\lambda/M)^2$. Further, define $f_\chi(z)$ by (3). Then $f_\chi(z)$ satisfies $f_\chi(-1/z) = f_\chi(z)$.

Remark. In case $\lambda = M$, $\varphi_\chi(z)$ coincides with $\eta_3(\chi\,;z)$ which was first defined in Katayama [1].

2. Lemmas. For any $y > 0$, we put

$$G(y) = -\{\log f_\chi(iy) + 2a\pi y\}.$$

Then from (1), we have

$$G(y) = T(\chi) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(m)a(n)}{m} \exp(-2mn\pi y/\lambda), \tag{4}$$

where $T(\chi)$ is the Gaussian sum defined by

$$T(\chi) = \sum_{h=0}^{M-1} \chi(h)\zeta_M^h.$$

Put

$$\xi(s) = T(\chi)\,(2\pi/\lambda)^{-s}\Gamma(s)\phi(s)L(s+1,\chi),$$

where $\Gamma(s)$ denotes the gamma function.

Lemma 1. *Let k be a real number. Then the next two conditions are equivalent.*

(A) $f_\chi(-1/z) = (z/i)^k f_\chi(z)$.

(B) *$\xi(s)$ can be continued through the whole s-plane as a meromorphic function satisfying $\xi(s) = \xi(-s)$ and*

$$\xi(s) + \frac{k}{s^2} + 2a\pi\left(\frac{1}{1+s} + \frac{1}{1-s}\right)$$

is entire and bounded in every vertical strip.

Proof. By (4) and Mellin's inversion formula, we obtain

(5) $$G(y) = \frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} \xi(s)\, y^{-s}\, ds,$$

where v is chosen large enough to be in the domain of absolute convergence of $\phi(s)$. Now assume (B). Then, shifting the line of integration in (5) to $Re(s) = -v$ and applying $\xi(s) = \xi(-s)$, we see that

(6) $$G(y) = G(1/y) + \frac{2a\pi}{y} - 2a\pi y + k\log y,$$

which yields

(7) $$\log f_\chi(i/y) = k\log y + \log f_\chi(iy).$$

Therefore

$$f_\chi(i/y) = y^k f_\chi(iy),$$

which is (A).

Next, we note that

$$\xi(s) = \int_0^\infty G(y) y^s d^\times y$$

for $Re(s)$ sufficiently large, where $d^\times y = \frac{dy}{y}$. It is easy to check that

$$\xi(s) = \int_1^\infty G(y) y^s d^\times y + \int_1^\infty G(1/y) y^{-s} d^\times y.$$

Assuming (A), we have (7) for any $y > 0$, so that we get (6) for any $y > 0$. Hence

$$\xi(s) + \frac{k}{s^2} + 2a\pi\left(\frac{1}{1+s} + \frac{1}{1-s}\right) = \int_1^\infty G(y)\,(y^s + y^{-s}) d^\times y.$$

Then the assertion (B) follows at once by noticing that $G(y) \ll \exp(-\pi y/\lambda)$ when $y \geqq 1$.

Lemma 2. *Let k be a real number. If* (2) *holds, then $\phi(s)$ satisfies the following four conditions.*

(a) *$\phi(s)$ can be continued through the whole s-plane as a meromorphic function.*

(b) *$s(s-1)\phi(s)L(s+1,\chi)$ is entire of finite order.*

(c) $(\lambda/M)^s\phi(s)L(-s,\chi) = \chi(-1)\,(\lambda/M)^{-s}\phi(-s)L(s,\chi)$.

(d) $\operatorname*{Res}_{s=0}\phi(s) = -\dfrac{k}{T(\chi)L(1,\chi)}$

and

$$\operatorname*{Res}_{s=1} \phi(s) = \frac{4a\pi^2}{\lambda T(\chi)L(2, \chi)}.$$

Proof. Noting that $\xi(s) = \xi(-s)$ is equivalent to (c) by the functional equation for $L(s, \chi)$, this follows easily from (B) of Lemma 1. So we omit the proof.

3. Proof of the theorem. We prove the first assertion. By our assumptions, $\phi(s)$ satisfies the four conditions of Lemma 2. Hence, putting $D(s) = \phi(s)/L(s, \chi)$, $D(s)$ can be continued through the whole s-plane as a meromorphic function of finite order and $(\lambda/M)^s D(s) = \chi(-1)(\lambda/M)^{-s} D(-s)$. Further, by (b), (c) and the assumption of $\phi(s)$, we see that $D(s)$ has a finite number of poles in the whole s-plane and $D(-s) = O(|(\lambda/M)^{2s}|)$ for $Re(s)$ sufficiently large. Then we can deduce from Lemma 5 in (2) that

$$D(s) = \sum_{m=1}^{K} c(m) m^{-s},$$

where K is the integral part of $(\lambda/M)^2$. By using the same argument as in the proof of Lemma 6 in [2], we find that $(\lambda/M)^2$ is an integer and

$$D(s) = \sum_{m \mid (\lambda/M)^2} c(m) m^{-s},$$

where $c(m)$, for m dividing $(\lambda/M)^2$, are integers such that $c(m) = \chi(-1)c((\lambda/M)^2/m)$ for any divisor m of $(\lambda/M)^2$. Therefore we get

$$\phi(s) = \Big(\sum_{m \mid (\lambda/M)^2} c(m) m^{-s}\Big) L(s, \chi). \tag{8}$$

Then it is easily verified that $\xi(s)$ is an integral function which is bounded in every vertical strip and satisfies $\xi(s) = \xi(-s)$. Hence, $a = k = 0$ and $f_\chi(z)$ is given by (3).

The remaining part of the theorem follows immediately from Lemma 1 since $\phi(s)$ is given by (8).

References

[1] K. Katayama: Zeta-functions, Lambert series and arithmetic functions. II. J. Reine Angew. Math., **268/269**, 251–270 (1974).

[2] M. Toyoizumi: On certain infinite products. II. Mathematika, **61**, 1–11 (1984).