82. On the Regularity of Prehomogeneous Vector Spaces

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Introduction. 0.1. Let G be a complex linear algebraic group, and $V(\neq 0)$ a complex vector space on which G acts via a rational representation. Let $V^\vee$ denote the dual space of V, on which G acts naturally.

0.2. If V has an open G-orbit, then (G, V) is called a *prehomogeneous vector space*. A prehomogeneous vector space (G, V) is called *regular* if there exists a (non-zero) relatively invariant polynomial function $f(x)=f(x_1,\cdots,x_n)$ on V such that

(0.3) $$\det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}\right) \not\equiv 0.$$

Let (G, V) be a regular prehomogeneous vector space, and take f so that (0.3) holds. Then the following assertions hold [5, §2].

(0.4) $(G, V^\vee)$ is also prehomogeneous.

(Moreover $(G, V^\vee)$. is regular.)

(0.5) Let O (resp. $O^\vee$) be the open G-orbit in V (resp. $V^\vee$). Then $O \simeq O^\vee$.

(0.6) Let $V\setminus O=(\cup_{i=1}^{l} S_i)\cup(\cup_{j=1}^{m} T_j)$ (resp. $V^\vee\setminus O^\vee=(\cup_{i=1}^{l'} S_i^\vee)\cup(\cup_{j=1}^{m'} T_j^\vee)$) be the irreducible decomposition, where the codimension of S_i and $S_i^\vee$ (resp. T_j and $T_j^\vee$) are one (resp. greater than one). Then $l=l'$.

Continue to assume (G, V) regular and prehomogeneous. Bearing (0.5) in mind, H. Yoshida posed the following problem.

Problem 1. $V\setminus O \simeq V^\vee \setminus O^\vee$?

Bearing (0.6) in mind, let us also consider the following problem.

Problem 2. *For any integer* $c>1$, card $\{j \mid \operatorname{codim} T_j=c\}$ = card $\{j \mid \operatorname{codim} T_j^\vee=c\}$?

We shall settle Problem 1 negatively in §1, and Problem 2 affirmatively in §2.

Convention. If no explanation is given, a lowercase letter should be understood as an element of the set denoted by the same uppercase letter. We define the Bruhat order of a Coxeter group so that the identity element is minimal. We denote the complex number field by $\boldsymbol{C}$, and the rational integer ring by $\boldsymbol{Z}$.

§1. 1.1. The following argument will be used. Let Z be a complex algebraic variety. We may assume that Z is defined over a field which is finitely generated over the rational number field. Then we can consider its specialization at enough general primes. Especially we obtain a variety over a finite field whenever the characteristic p and the cardinality q of the field are large enough. Let $|Z|=|Z|(q)$ be the cardinality of the rational points of the variety obtained above. We understand that $|Z|$ is a function of q, and defined if p and q are large enough. We write $|Z| = |Z'|$ etc., if equality holds for large p and q. If $Z \simeq Z'$, then $|Z| = |Z'|$.

1.2. Now we take up again the prehomogeneous vector space studied in [2]. Let $V = V^\vee = M_n$ be the totality of $n \times n$-matrices. Define the $GL_n \times GL_n$-action on V (resp. $V^\vee$) by $(g_1, g_2) \cdot v = g_1 v g_2^{-1}$ (resp. $(g_1, g_2) \cdot v^\vee = g_2 v^\vee g_1^{-1}$) for $g_i \in GL_n$. We consider $(GL_n \times GL_n, V^\vee)$ as the dual of $(GL_n \times GL_n, V)$ by the pairing $\langle v^\vee, v\rangle = \mathrm{trace}(v^\vee v)$. Let W be the totality of permutation matrices in GL_n, which we identify with the symmetric group. Let s_i be the transposition $(i, i+1)$ and $S = \{s_1, \ldots, s_{n-1}\}$. For $I \subset S$, let W_I be the subgroup of W generated by I, w_I the longest element of W_I, and $I' := w_S I w_S$. (The pair (W, S) is a Coxeter system. For its generality, we refer to [1, Chapter 4].) Put $P_I = BW_IB$ (the standard parabolic subgroup of type I). For $x = (x_{ij}) \in M_n$, put $f_k(x) = \det(x_{n-k+i,j})_{1 \le i,j \le k}$. Then $f_k^{-1}(0) = \overline{Bw_S s_k B}$, where the (Zariski) closure is taken in M_n.

1.3. Let us consider the prehomogeneous vector space $(P_{I'} \times P_J, V)$ and its dual $(P_{I'} \times P_J, V^\vee)$, which are regular since f_n satisfies (0.3). Let O (resp. $O^\vee$) be the open orbit in V (resp. $V^\vee$). Then $O = P_{I'} w_S P_J = Bw_S W_I W_J B$ and $O^\vee = BW_J W_I w_S B$ (cf. [1, no.2.1, Lemma 1]). Let X be the set of minimal elements of $W \setminus W_I W_J$ with respect to the Bruhat order. Then the irreducible components of $V \setminus O$ (resp. $V^\vee \setminus O^\vee$) are $\{f_n^{-1}(0), \overline{Bw_S xB}\ (x \in X)\}$ (resp. $\{f_n^{-1}(0), \overline{Bx^{-1}w_S B}\ (x \in X)\}$).

Lemma 1.4. *$f_k^{-1}(0) \not\simeq f_l^{-1}(0)$ unless $k = l$.*

Proof. Use (1.1), noting $|M_n| - |f_k^{-1}(0)| = |M_n \setminus f_k^{-1}(0)| = q^{n^2-k^2} \Pi_{i=0}^{k-1}(q^k - q^i)$.

Example 1.5. Let $I = J = S \setminus \{s_k\}$. Then $X = \{s_k\}$, and hence $\overline{Bw_S s_k B} = f_k^{-1}(0)$ (resp. $\overline{Bs_k w_S B} = \overline{Bw_S s_{n-k} B} = f_{n-k}^{-1}(0)$) and $f_n^{-1}(0)$ are the irreducible components of $V \setminus O$ (resp. $V^\vee \setminus O^\vee$). Hence $V \setminus O \not\simeq V^\vee \setminus O^\vee$ unless $k = n - k$.

Example 1.6. Let $n = 4$, $I = \{s_2\}$ and $J = \{s_1\}$. Then $X = \{s_3, s_1 s_2\}$, and hence $V \setminus O$ (resp. $V^\vee \setminus O^\vee$) has two irreducible components $\{f_4^{-1}(0), f_3^{-1}(0)\}$ (resp. $\{f_4^{-1}(0), f_1^{-1}(0)\}$) of condimension one, and one component $C := \overline{Bw_S s_1 s_2 B} = \{(x_{ij}) \mid x_{41} = x_{42} = 0\}$ (resp. $C^\vee := \overline{Bw_S s_2 s_3 B} = \{(x_{ij}) \mid \mathrm{rank}\begin{pmatrix} x_{31} x_{32} x_{33} \\ x_{41} x_{42} x_{43}\end{pmatrix} < 2\}$) of condimension two. Since $|C| = q^{14}$ and $|C^\vee| = q^{16} - q^{10}(q^3 - 1)(q^3 - q)$, $C \not\simeq C^\vee$. Thus the higher codimensional part of $V \setminus O$ and $V^\vee \setminus O^\vee$ can also become non-isomorphic.

Remark 1.7 (cf. [4, §4, Remark 26] and [2]). The following conditions are equivalent.

(1) O is an affine variety.

(2) $GL_n \setminus O$ is a hypersurface of GL_n.

(3) $W \setminus w_S W_I W_J = \cup_{s \in S \setminus I \cup J} \{w \in W \mid w \le w_S s\}$.

(4) $W_I W_J = W \setminus \cup_{s \in S \setminus I \cup J} \{w \in W \mid w \le w_S s\}$.

(5) $W_I W_J = W_{I \cup J}$.

(6) Let $\{I_\alpha\}_\alpha$ (resp. $\{J_\beta\}_\beta$) be the connected components of the Dynkin diagram (= the Coxeter diagram) of I (resp. J). If the Dynkin diagram of $I_\alpha \cup J_\beta$ is connected, then $I_\alpha \subset J_\beta$ or $I_\alpha \supset J_\beta$.

Proof. Since $\{w \in W \setminus w_S W_I W_J \mid l(w) = l(w_S) - 1\} = \{w_S s \mid s \in S \setminus I \cup J\}$, we get (2) $\Leftrightarrow$ (3). Let us prove (5) $\Rightarrow$ (6). Assume that $I_\alpha \setminus J_\beta \neq \phi$, $J_\beta \setminus I_\alpha \neq \phi$, and the Dynkin diagram of $I_\alpha \cup J_\beta$ is connected. Then we may assume that the Dynkin diagram of $I_\alpha \cup J_\beta =: \{s_k, \cdots, s_l\}$ is $s_k—s_{k+1} \cdots —s_l$, $s_k \in J_\beta \setminus I_\alpha$, and $s_l \in I_\alpha \setminus J_\beta$. Using [1, no. 1.5, Lemma 4], we can show that $w := s_k s_{k+1} \cdots s_l$ is the unique reduced expression of w. Hence $w \notin W_I W_J$, although $w \in W_{I \cup J}$. Thus we get the implication. The remainder is obvious or explained in [2].

§2. Here we give an affirmative answer to Problem 2. (See (2.2).) All that is necessary in our argument is the isomorphism $O \simeq O^\vee$ as abstract varieties. Therefore the answer remains affirmative even if 'regular' is replaced with 'quasi-regular' [5].

2.1. Let V and $V^\vee$ be complex vector spaces of dimension n, $O \subset V$ (resp. $O^\vee \subset V^\vee$) a Zariski open set, $\{S_1, \cdots, S_l\}$ (resp. $\{S_1^\vee, \cdots, S_{l'}^\vee\}$) the irreducible components of $V \setminus O$ (resp. $V^\vee \setminus O^\vee$) of codimension one, and $\{T_1, \cdots, T_m\}$ (resp. $\{T_1^\vee, \cdots, T_{m'}^\vee\}$) those of codimension greater than one. Put $\Omega = V \setminus \cup_i S_i$ and $\Omega^\vee = V \setminus \cup_i S_i^\vee$. Let $f_i = 0$ (resp. $f_i^\vee = 0$) be a defining equation of S_i (resp. $S_i^\vee$). Let $\langle f_1, \cdots, f_l \rangle$ be the multiplicative group generated by $\{f_1, \cdots, f_l\}$. Define $\langle f_1^\vee, \cdots, f_{l'}^\vee \rangle$ similarly.

Proposition 2.2. *If $O \simeq O'$, then this isomorphism extends to $\Omega \simeq \Omega^\vee$, which induces a bijection $\{T_1, \cdots, T_m\} \to \{T_1^\vee, \cdots, T_{m'}^\vee\}$ and an isomorphism $\langle f_1, \cdots, f_l \rangle \to \langle f_1^\vee, \cdots, f_{l'}^\vee \rangle$. Especially $l = l'$ and* $\operatorname{card}\{j \mid \operatorname{codim} T_j = c\} = \operatorname{card}\{j \mid \operatorname{codim} T_j^\vee = c\}$ *for any $c > 1$.*

Proof. The isomorphism $O \simeq O^\vee$ induces $\Gamma(O, \mathcal{O}) \simeq \Gamma(O^\vee, \mathcal{O})$, where $\mathcal{O}$ denotes the sheaf of regular functions on the respective variety. Since $\Omega \setminus O$ (resp. $\Omega^\vee \setminus O^\vee$) is of codimension greater than one in Ω (resp. $\Omega^\vee$), $\Gamma(O, \mathcal{O}) = \Gamma(\Omega, \mathcal{O})$ (resp. $\Gamma(O^\vee, \mathcal{O}) = \Gamma(\Omega^\vee, \mathcal{O})$). Hence the isomorphism $O \simeq O^\vee$ extends to $\Omega \simeq \Omega^\vee$. Note that the irreducible components of $\Omega \setminus O = T_1 \cup \cdots \cup T_m \setminus S_1 \cup \cdots \cup S_l$ are naturally in one to one correspondence with $\{T_1, \cdots, T_m\}$, and similar for $\Omega^\vee \setminus O^\vee$. Hence the isomorphism $\Omega \setminus O \simeq \Omega^\vee \setminus O^\vee$ induces a bijection $\{T_1, \cdots, T_m\} \to \{T_1^\vee, \cdots, T_{m'}^\vee\}$. Using the long exact sequence of $\boldsymbol{Z}$-cohomologies with compact support

$$0 = H_c^{2n-2}(V) \to H_c^{2n-2}(S_1 \cup \cdots \cup S_l) \to H_c^{2n-1}(\Omega) \to H_c^{2n-1}(V) = 0,$$

we get the natural isomorphism $\langle f_1, \cdots, f_l \rangle \simeq H_c^{2n-1}(\Omega)$, and similarly $\langle f_1^\vee, \cdots, f_{l'}^\vee \rangle \simeq H_c^{2n-1}(\Omega^\vee)$. Thus $\Omega \simeq \Omega^\vee$ induces the desired isomorphism. (Note that our argument works also in the positive characteristic case if H_c^* is understood as an l-adic étale cohomology.)

Remark 2.3. In the above proposition, it is enough to assume the existence of a homeomorphism $\varphi : \Omega \to \Omega^\vee$ such that $\varphi(O) = O^\vee$. (Assume that an analytic space Z is locally Euclidean at $z \in Z$. Then the germ of analytic space (Z, z), is of pure cohomological dimension, and hence the number of local irreducible components at z is rank $(\varinjlim H_c^{top}(U)) = 1$, where U runs over the open neighbourhoods of z. Hence the irreducible components of Z are the closures of the connected components of the locus where Z is locally Euclidean. Thus irreducible components are characterized topologically, and hence we get the bijection $\{T_1, \cdots, T_m\} \to \{T_1^\vee, \cdots, T_{m'}^\vee\}$.)

In the case where $m = m' = 0$, it is enough to assume the existence of a continuous mapping $O \to O^\vee$ inducing a quasi-isomorphism $R\Gamma_c(O, \boldsymbol{Z}) \to R\Gamma_c(O^\vee, \boldsymbol{Z})$. Cf. (2.5) below.

Remark 2.4. Assume the reductivity of G instead of the regularity of (G, V). As the argument of [4, p.71] shows, (0.4)-(0.6) remain valid, and the answer to both problems is affirmative.

If we do not assume the regularity nor the reductivity, then the following example settle everything negatively. Let G_i and $V_i (i = 1, 2)$ be as in (0.1), and $H = \mathrm{Hom}_{\boldsymbol{C}}(V_1, V_2^\vee)$ (additive group). Define the semi-direct product $G := (G_1 \times G_2) \ltimes H$ so that $(g_1, g_2)h(g_1, g_2)^{-1} = g_2 h g_1^{-1}$, and define its action on $V := V_1 \oplus V_2^\vee$ by $(g_1, g_2) \cdot (v_1, v_2^\vee) = (g_1 v_1, g_2 v_2^\vee)$ and $h \cdot (v_1, v_2^\vee) = (v_1, h(v_1) + v_2^\vee)$. Then we can show that (G, V) (resp. $(G, V^\vee)$) is prehomogeneous if and only if (G_1, V_1) (resp. (G_2, V_2)) is prehomogeneous. Assume the prehomogeneity of both (G_i, V_i). Let O_i be the respective open orbit. Then the open orbit of (G, V) (resp. $(G, V^\vee)$) is $O_1 \times V_2^\vee$ (resp. $V_1^\vee \times O_2$).

Example 2.5. Let $V_1 = M_2 (= \boldsymbol{C}^2 \otimes \boldsymbol{C}^2)$ and V_2 be the third symmetric tensor $S^3(\boldsymbol{C}^2)$ of $\boldsymbol{C}^2$. Then $G_1 = G_2 = GL_2$ acts on V_1 by the left multiplication and on V_2 naturally. From these two prehomogeneous vector spaces $(G_i, V_i) (i = 1,2)$, we can construct prehomogeneous vector spaces (G, V) and $(G, V^\vee)$ as in (2.4), whose open orbits are $O := O_1 \times V_2^\vee$ and $O^\vee := V_1^\vee \times O_2$. Fix a point $o_2 \in O_2$. Then the morphism $O_1 = GL_2 \ni g \to go_2 \in O_2$ and any linear isomorphism $V_2^\vee \to V_1^\vee$ induce a morphism $O \to O^\vee$ which induces a quasi-isomorphism $R\Gamma_c(O, \boldsymbol{Z}) \to R\Gamma_c(O^\vee, \boldsymbol{Z})$.

Remark 2.6. Assume the existence of an isomorphism between the open orbits of two prehomogeneous vector spaces, say $(G_i, V_i) (i = 1,2)$, as abstract varieties. The author does not know whether this assumption implies the existence of a linear isomorphism between (V_i, Ω_i)'s, where Ω_i's are defined as in (2.1) using the open orbits $O_i \subset V_i$.

Remark 2.7. In [3], prehomogeneous vector spaces with non-reductive groups play an important role. Most of them are not regular.

References

[1] N. Bourbaki: Groupes et algèbres de Lie, chapitres 4,5 et 6. Masson (1981).

[2] A. Gyoja: A counterexample in the theory of prehomogeneous vector spaces. Proc. Japan Acad., **66A**, 26–27 (1990).

[3] ——: Highest weight modules and b-functions of semi-invariants. I (preprint).

[4] M. Sato and T. Kimura: A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J., **65**, 1–155 (1977).

[5] M. Sato, T. Shintani and M. Muro: Theory of prehomogeneous vector spaces (algebraic part). The English translation of Sato's lecture from Shintani's note. Nagoya Math. J., **120**, 1–34 (1990).