# 79. On the Starlikeness of the Alexander Integral Operator

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**Abstract** : Denote by $A$ the class of functions $f(z)$ analytic in the unit disk $D$ and normalised so that $f(0) = f'(0) - 1 = 0$. For $f(z) \in A$, let $F(z) = \int_0^z [f(t)/t]\,dt$ for $z \in D$. We find estimate on $\beta$ so that $\mathrm{Re} f'(z) > -\beta$ will ensure the starlikeness of $F(z)$. Our conclusion improves the well-known results.

**1. Introduction.** Denote by $A$ the class of functions $f(z)$ which are analytic in the unit disc $D = \{z : |z| < 1\}$ and normalised so that $f(0) = f'(0) - 1 = 0$. Let $R_\alpha$ be the subclass of $A$ satisfying $\mathrm{Re} f'(z) > \alpha$ for $z \in D$ and $S^*$ be the subset of starlike functions, i. e.

$$S^* = \{f(z) \in A : \mathrm{Re}[zf'(z)/f(z)] > 0 \text{ for } z \in D\}.$$

For $f(z) \in A$, let

$$F(z) = \int_0^z [f(t)/t]\,dt \quad z \in D. \tag{1}$$

This integral operator was first introduced by J. W. Alexander. In paper [1], R. Singh and S. Singh showed that if $f(z) \in R_0$, then $\mathrm{Re}[F(z)/z] > 1/2$ $(z \in D)$, and if $\mathrm{Re} f'(z) > -1/4$, then $F(z) \in S^*$. Recently M. Nunokawa and D. K. Thomas [2] improved the second result by showing that if $\mathrm{Re} f'(z) > -0.262$, then $F(z) \in S^*$.

In this paper we will improve both two conclusions.

**2. Results and proofs.** In proving our results, we need the following lemmas.

**Lemma 1** ([3]) . *Let $f(z)$ be analytic and $g(z)$ convex in $D$ (that is, in $D$, $g(z)$ satisfies $\mathrm{Re}[1 + zg''(z)/g'(z)] > 0$). If $f(z) \prec g(z)$ $(z \in D)$, then we have*

$$z^{-1}\int_0^z f(t)\,dt \prec z^{-1}\int_0^z g(t)\,dt,$$

*where "$\prec$" denotes the subordination.*

**Lemma 2** ([4]) . *If $g(z) \in K$— the normalised class of convex functions, then*

$$G(z) = \frac{2}{z}\int_0^z g(t)\,dt \in K.$$

**Lemma 3** ([5]). *Let $w(z)$ be a non-constant regular function in $D$, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at $z_0$, then we have $z_0 w'(z_0) = kw(z_0)$, where $k$ is a real number,*

$k \geqslant 1$.

**Theorem 1.** *Let* $f(z) \in R_\alpha$, *then*

$$\mathrm{Re}[F(z)/z] > 2\alpha - 1 + (1-\alpha)\frac{\pi^2}{6} \quad (z \in D),$$

*where* $F(z)$ *is defined by* (1).

*Proof.* From the definition of $F(z)$, we have

(2) $$F'(z) + zF''(z) = f'(z)$$

for $f(z) \in R_\alpha$, we have

$$F'(z) + zF''(z) = f'(z) \prec \frac{1 + (1-2\alpha)z}{1-z} = p_\alpha(z) \quad (z \in D).$$

It is easy to know $p_\alpha(z)$ is convex in $D$, so using Lemma 1, we get

$$z^{-1}\int_0^z [F'(t) + tF''(t)]dt \prec z^{-1}\int_0^z p_\alpha(t)dt,$$

that is,

(3) $$F'(z) \prec 2\alpha - 1 - \frac{2(1-\alpha)}{z}\log(1-z) = \varphi_1(z).$$

It is easy to know that if $f(z) \prec g(z)$, then $af + b \prec ag + b$ ($a$, $b$ are constants and $a \neq 0$) too, and if $f$ is convex in $D$, then $af + b$ ($a$, $b$ are constants and $a \neq 0$) is convex in $D$ too. So from Lemma 2, we know that $\varphi_1(z)$ is convex in $D$. Applying Lemma 1 to (3), we obtain

$$F(z)/z \prec z^{-1}\int_0^z \varphi_1(t)dt = \varphi_2(z).$$

So we have

$$\mathrm{Re}[F(z)/z] > \min_{|z| \leqslant r} \mathrm{Re}[\varphi_2(z)] \quad (|z| \leqslant r).$$

As indicated above, $\varphi_2(z)$ is convex in $D$, and it is easy to check that $\overline{\varphi_2(\bar{z})} = \varphi_2(z)$, so $\varphi_2(z)$ maps $|z| \leqslant r$ onto a convex region which is symmetric with respect to the real axis. Comparing $\varphi_2(r)$ and $\varphi_2(-r)$ we know

$$\min_{|z| \leqslant r} \mathrm{Re}[\varphi_2(z)] = \varphi_2(-r) = \frac{1}{r}\int_0^r \left[2\alpha - 1 + \frac{2(1-\alpha)}{t}\log(1+t)\right] dt.$$

Similarly $\varphi_2(z)$ maps $D$ onto a convex region which is symmetric with respect to the real axis, so we get

$$\mathrm{Re}[F(z)/z] > \int_0^1 \left[2\alpha - 1 + \frac{2(1-\alpha)}{t}\log(1+t)\right] dt \quad (z \in D).$$

Expanding the integrand into Taylor series about $t$ and integrating it, we can obtain

$$\begin{aligned} \mathrm{Re}[F(z)/z] &> 2\alpha - 1 + 2(1-\alpha)\cdot\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} \\ &= 2\alpha - 1 + (1-\alpha)\cdot\frac{\pi^2}{6}. \end{aligned}$$

The proof of the theorem is completed.

If we let $\alpha = 0$, we have the following corollary.

**Corollary 1.** *Let* $f(z) \in R_0$, *then*

$$\mathrm{Re}[F(z)/z] > \frac{\pi^2}{6} - 1 = 0.6449\cdots \quad (z \in D).$$

*The constant* $\frac{\pi^2}{6} - 1$ *cannot be replaced by any larger one.*

The second assertion can be seen from the function:

$$f(z) = -z - 2\log(1-z) \in R_0.$$

**Remark.** This corollary improves and sharpens the corresponding result of R. Singh and S. Singh [1].

**Theorem 2.** *Suppose that* $f(z) \in A$ *and* $F(z)$ *is given by* (1). *If* $\mathrm{Re} f'(z) > -\beta = \dfrac{6-\pi^2}{24-\pi^2} = 0.2738\cdots$ $(z \in D)$, *then* $F(z) \in S^*$.

*Proof.* First we prove that if $f(z)$ satisfies the hypothesis of the theorem, then we have $\mathrm{Re} F'(z) > 0 (z \in D)$, thus $F(z)$ is univalent in $D$. In fact, from the condition and the definition of $F(z)$ we have

$$\frac{F'(z) + zF''(z) + \beta}{1+\beta} = \frac{f'(z)+\beta}{1+\beta} \prec \frac{1+z}{1-z},$$

using Lemma 1 we obtain

$$F'(z) \prec (1+\beta)\left[-1 - \frac{2}{z}\log(1-z)\right] - \beta. \tag{4}$$

So

$$\begin{aligned}\mathrm{Re}F'(z) &> (1+\beta)\inf_{|z|<1}\mathrm{Re}\left[-1-\frac{2}{z}\log(1-z)\right] - \beta \\ &= (1+\beta)(-1+2\log 2) - \beta > 0 \qquad (z \in D).\end{aligned}$$

Second we estimate the lower bound of $\mathrm{Re}[F(z)/z]$. Since the function on the right-hand side of (4) is convex, using Lemma 1 again we get

$$F(z)/z \prec (1+\beta)\frac{1}{z}\int_0^z \left[-1-\frac{2}{t}\log(1-t)\right]dt - \beta,$$

thus

$$\mathrm{Re}[F(z)/z] > (1+\beta)\left(\frac{\pi^2}{6} - 1\right) - \beta = 2\beta \ (z \in D). \tag{5}$$

Now we can prove $F(z) \in S^*$. Let

$$[zF'(z)]/F(z) = [1+w(z)]/[1-w(z)]. \tag{6}$$

Since $F(z)$ is univalent in $D$, $w(z)$ defined in (6) is analytic in $D$ and $w(0) = 0$, $w(z) \neq 1$. From (6) we have

$$F'(z) + zF''(z) = \frac{F(z)}{z}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^2 + \frac{2zw'(z)}{(1-w(z))^2}\right]. \tag{7}$$

We can claim that $|w(z)| < 1$ in $D$. In fact, if not, there exists a point $z_0 \in D$ such that $\max_{|z|\leqslant|z_0|}|w(z)| = |w(z_0)| = 1$, then from Lemma 3 we have $z_0 w'(z_0) = kw(z_0) = ke^{i\theta}$ for $0 < \theta < 2\pi$ where $k \geqslant 1$. With $z = z_0$, it follows from (7) that

$$\begin{aligned}\mathrm{Re}[F'(z) + zF''(z)] &= \mathrm{Re}\left\{\frac{F(z_0)}{z_0}\left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right]\right\} \\ &= -\frac{1+\cos\theta+k}{1-\cos\theta}\mathrm{Re}\left\{\frac{F(z_0)}{z_0}\right\} \\ &\leq -\beta,\end{aligned} \tag{8}$$

where we used the inequality (5). From the definition of $F(z)$ and (8) we have $\mathrm{Re} f'(z_0) \leq -\beta$, which contradicts our hypothesis, so we have $|w(z)| < 1$ in $D$. Hence from (6) we know $\mathrm{Re}[zF'(z)/F(z)] > 0$ $(z \in D)$, which means $F(z) \in S^*$.

**Remark.** For $\beta = 0.2738\cdots > 0.262 > 1/4$, so Theorem 2 is the improvement of the corresponding results obtained by [1] and [2].

**Corollary 2.** *Let* $g(z) \in A$ *and* $G(z)$ *be defined by* $zG'(z) = \int_0^z [g(t)/t]\, dt$. *If* $\mathrm{Re}\, g'(z) > -\beta$, $(z \in D)$, *then* $G(z) \in K$ *where* $\beta$ *is defined in Theorem* 2.

## References

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