48. The Variance of the Single Point Range of Two Dimensional Recurrent Random Walk

By Yuji HAMANA

Department of Applied Science, Faculty of Engineering, Kyushu University

(Communicated by Kiyosi ITÔ, M. J. A., Sept. 14, 1992)

§ 1. **Introduction and results.** Let $\{X_n\}_{n=1}^{\infty}$ be the sequence of $\boldsymbol{Z}^d$ valued independent identically distributed random variables defined on a probability space $(\Omega, \mathfrak{B}, P)$. Define $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$. The sequence of these random variables $\{S_n\}_{n=0}^{\infty}$ is called a random walk starting at 0. Let Q_n be the number of the distinct lattice points which the random walk visits once and only once in the first n steps. This random variable is called the single point range of the random walk up to time n or merely the single point range.

We assume the random walk is aperiodic, that is, no proper subgroup of the state space contains the set of x such that $P(X_1 = x) > 0$. If there exists a positive integer N_x satisfying that $P(S_n = x) > 0$ whenever $n \geq N_x$ for every $x \in \boldsymbol{Z}^d$, the random walk is called strongly aperiodic.

We obtain the asymptotic behavior of the variance of Q_n and then can show immediately that the weak law of large number is obeyed.

Theorem A. *For a strongly aperiodic, two dimensional random walk with* $EX_1 = 0$ *and* $E|X_1|^2 < \infty$, *there exists a positive constant* K *such that*

$$\lim_{n\to\infty} \frac{(\log n)^6 Var\, Q_n}{n^2} = c_1^4 K,$$

where $c_1 = 2\pi(\det \Sigma)^{\frac{1}{2}}$ *and* Σ *is the covariance matrix of* X_1.

Theorem B. *Under the same condition as in Theorem A, it holds that*

$$\lim_{n\to\infty} P(|Q_n - EQ_n| > \varepsilon EQ_n) = 0$$

for any $\varepsilon > 0$.

By the following theorems, we aware that the estimate of $Var\, Q_n$ is different from that in the case $d \geq 3$. Let $p = P(S_n \neq 0\ n = 1,2, \cdots)$.

Theorem 1 (Hamana [2]). *If* $d \geq 4$ *and* $p < 1$, *then there exists a positive constant* σ^2 *such that*

$$\lim_{n\to\infty} \frac{Var\, Q_n}{n} = \sigma^2.$$

Theorem 2 (Hamana). *If* $d = 3$ *and* $p < 1$, *then there exists a slowly varying function* $\psi(n)$ *such that*

$$\lim_{n\to\infty} \frac{Var\, Q_n}{n\psi(n)} = 1.$$

Let R_n be the number of the distinct sites visited at least once by a random walk. In the two dimensional case, the asymptotic behavior of R_n was derived by Jain and Pruitt. However, this is not similar to $Var\, Q_n$.

Theorem 3 (Jain-Pruitt [4]). *For an aperiodic, two dimensional random walk with* $EX_1 = 0$ *and* $E|X_1|^2 < \infty$, *there exists a positive constant* L *such that*

$$\lim_{n\to\infty} \frac{(\log n)^4 Var\, R_n}{n^2} = c_1^2 L,$$

where c_1 *is the same as in Theorem A.*

§ 2. Sketch of proofs. We will introduce some notations. For $x \in \boldsymbol{Z}^2$, τ_x will denote the first hitting time of x, i.e.

$$\tau_x = \inf\{n \geq 1\,;\, S_n = x\}\,;$$

if there are no integers satisfying $S_n = x$, then $\tau_x = \infty$. We will use for $f_n = P(\tau_0 = n)$ and $r_n = \sum_{k=n+1}^{\infty} f_k$.

The following lemma was established in Jain and Pruitt [4].

Lemma. *For a strongly aperiodic, two dimensional random walk with* $EX_1 = 0$ *and* $E|X_1|^2 < \infty$,

$$f_n = \frac{c_1}{n(\log n)^2} + o\left(\frac{1}{n(\log n)^2}\right),$$

where c_1 *is the same as in the statement of Theorem A.*

By using this Lemma, we can obtain

$$r_n = \frac{c_1}{\log n} + o\left(\frac{1}{\log n}\right).$$

Proof of Theorem A. We have

$$\begin{aligned} Var\, Q_n = {} & 2\left\{\sum_{j=1}^{n}\sum_{i=1}^{j-1} r_{n-j}r_{n-i}r_i(r_i - r_j) - \sum_{i=1}^{n}\sum_{j=1}^{n-i}\sum_{\mu=1}^{i} f_{i+\mu}r_{n-j-i}^2 r_{i-\mu}\right\} \\ & + 2\left(4K_1 + \frac{1}{2} - \frac{1}{6}\pi^2\right)\frac{c_1^4 n^2}{(\log n)^6} \\ & + o\left(\frac{n^2}{(\log n)^6}\right), \end{aligned}$$

where

$$K_1 = -\int_0^1 \frac{\log x}{1 - x + x^2}\,dx = 1.17195361935\cdots.$$

This estimate is highly non trivial. However, we will omit the proof in this paper. We will give the complete proof in a forthcoming paper.

Since

$$\begin{aligned} & \sum_{j=1}^{n}\sum_{i=1}^{j-1} r_{n-j}r_{n-i}r_i(r_i - r_j) \\ & \qquad = r_n^2 \sum_{j=1}^{n}\sum_{i=1}^{j-1} r_{n-i}(r_i - r_j) + \frac{3c_1^4 n^2}{2(\log n)^6} + o\left(\frac{n^2}{(\log n)^6}\right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=1}^{n}\sum_{j=1}^{n-i}\sum_{\mu=1}^{i} f_{i+\mu}r_{n-j-i}^2 r_{i-\mu} \\ & \qquad = r_n^2 \sum_{j=1}^{n}\sum_{i=1}^{j-1} r_{n-j}(r_i - r_j) + \frac{3c_1^4 n^2}{2(\log n)^6} + o\left(\frac{n^2}{(\log n)^6}\right), \end{aligned}$$

the first term of the right hand side is asymptotically equal to

$$-2r_n^2 \sum_{j=1}^{n}\sum_{i=1}^{j-1} (r_{n-j} - r_{n-i})(r_i - r_j) = -\frac{c_1^4 n^2}{(\log n)^6} + o\left(\frac{n^2}{(\log n)^6}\right).$$

Hence it holds that

$$Var\, Q_n = \frac{c_1^4 K n^2}{(\log n)^6} + o\left(\frac{n^2}{(\log n)^6}\right),$$

where $K = 2(4K_1 - \frac{1}{6}\pi^2) = 3.0857608\cdots$. Q.E.D.

Proof of Theorem B. By using the asymptotic behavior of r_n, we can obtain

$$EQ_n = \frac{c_1^2 n}{(\log n)^2} + o\left(\frac{n}{(\log n)^2}\right).$$

By Chebyshev's inequality,

$$P(|Q_n - EQ_n| > \varepsilon EQ_n) \leq \frac{Var\, Q_n}{\varepsilon^2 (EQ_n)^2} = O\left(\frac{1}{(\log n)^2}\right)$$

for any $\varepsilon > 0$. Q.E.D.

Remark. In Jain and Pruitt [4], the strong law of large number is established. To prove this limit theorem, the monotonicity of R_n with respect to n is quite important. Since Q_n does not have such property, we can show only the weak law.

For the fluctuation of R_n, Le Gall showed the central limit theorem.

Theorem 4 (Le Gall [5]). *For an aperiodic, two dimensional random walk with $EX_1 = 0$ and $E|X_1|^2 < \infty$,*

$$\lim_{n\to\infty} \frac{(\log n)^2 (R_n - ER_n)}{n} = -4\pi^2 (\det \Sigma) \gamma(\mathscr{C}) \text{ in law,}$$

where

$$\gamma(\mathscr{C}) = \int\int_C \delta_0(W_s - W_t)\,ds\,dt - E\left[\int\int_C \delta_0(W_s - W_t)\,ds\,dt\right],$$

$\mathscr{C} = \{(s, t) \in \boldsymbol{R}^2; 0 \leq s < t \leq 1\}$, *and* $\{W_t\}_{t\geq 0}$ *is a two dimensional Brownian motion.*

However, this type of theorem of Q_n remains open.

References

[1] P. Erdős and S. J. Taylor: Some problems concerning the structure of random walk paths. Acta. Math. Sci. Hungar., **11**(1960).

[2] Y. Hamana: On the central limit theorem for the multiple point range of random walk. J. Fac. Sci. Univ. Tokyo, **39**, no. 2 (1992).

[3] N. C. Jain and W. E. Pruitt: The range of recurrent random walk in plane. Z. Wahr. Verw. Geb., **16** (1970).

[4] ——: The range of random walk. Proc. Sixth Berkeley Symp. on Math. Stat. and Prob., Berkeley, University of California Press (1973).

[5] J. F. Le Gall: Propriétés d'intersection des marches aléatoires. Comm. Math. Phys., **104**(1986).

[6] F. Spitzer: Principles of random walk. Graduate Texts in Mathematics. vol. 34, Springer-Verlag (1976).