38. *Singular Variation of Non-linear Eigenvalues*

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1. Introducrion. Recently several papers have appeared concerning semi-linear elliptic boundary value problems. See, for example, Dancer [1], Lin [3], Wang [7] and the literatures cited there.

We consider the following problem. Let M be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary ∂M. Let w be a fixed point in M. Removing an open ball $B(\varepsilon;w)$ of radius ε with the center w from M, we get $M_\varepsilon = M\backslash\overline{B(\varepsilon;w)}$. We consider the minimizing problem $(1.1)_\varepsilon$ for $\varepsilon>0$. Fix $p>1$. We put

$$\lambda(\varepsilon)=\inf_{X_\varepsilon}\int_{M_\varepsilon}|\nabla u|^2dx, \tag{$(1.1)_\varepsilon$}$$

where $X_\varepsilon=\{u\in H_0^1(M_\varepsilon),\ \|u\|_{L^{p+1}(M_\varepsilon)}=1\}$. We consider the asymptotic behaviour of $\lambda(\varepsilon)$ as ε tends to 0. It is well known that there exists at least one positive solution u_ε which attains $(1.1)_\varepsilon$ in case of $p\in(1,5)$. We know that the minimizer satisfies $-\Delta u_\varepsilon=\lambda(\varepsilon)u_\varepsilon^p$ in M_ε and $u_\varepsilon=0$ on ∂M_ε. we put

$$\lambda=\inf_X\int_M|\nabla u|^2dx,$$

where $X=\{u\in H_0^1(M),\ \|u\|_{L^{p+1}(M)}=1\}$.

We have the following

Theorem. *Assume that the positive solution of* $-\Delta \boldsymbol{u}=\lambda\boldsymbol{u}^p$ *in* M *under the Dirichlet condition on* ∂M *is unique. Assume also that the ground state solution* u_ε *for* $(1.1)_\varepsilon$ *is unique for any small* $0<\varepsilon\ll1$. *We assume that* $\mathrm{Ker}\,(\Delta+\lambda(\varepsilon)pu_\varepsilon^{p-1})=\{0\}$ *for* $0<\varepsilon\ll1$. *Here* u_ε *is the positive minimizer of* $(1.1)_\varepsilon$. *Then,*

$$\lambda(\varepsilon)-\lambda=4\pi\varepsilon u(w)^2+o(\varepsilon) \tag{1.2}$$

holds for $p\in(1,2)$. *Here* u *is the minimizer with respect to* λ.

Remarks. We do not treat the case $p=1$ here. In fact, if $p=1$, then $\lambda(\varepsilon)$, (λ, respectively) is the first eigenvalue of $-\Delta$ in M_ε (M, respectively) under the Dirichlet condition and we have an analogous result of (1.2). See [6]. The author wanted to generalize the asymptotic formula for $p=1$ to other cases. This is a motivation of our research.

The domain M such that the number of positive solution of $-\Delta\boldsymbol{u}=\lambda\boldsymbol{u}^p$ in M under the Dirichlet condition on ∂M is exactly one is given by Dancer [1], Gidas-Ni-Nirenberg [2].

The author does not know any example of a domain which satisfies the first, the second and the third assumptions in the Theorem. Even if M is a ball with the center w, the author can not prove that the second and the third assumptions hold. However, the author believes that if the first assumption of the Theorem is satisfied, then the second and the third assumptions hold generically in some sense.

The condition that $\mathrm{Ker}\,(\varDelta+\lambda(\varepsilon)pu_\varepsilon^{p-1})=\{0\}$, which is a sufficient condition for the existence of $\lambda'(\varepsilon)$, can not be excluded at the present time.

The author restricts himself to the case of the Dirichlet boundary condition in this paper. The case of the Robin condition is discussed in [5].

2. Outline of our proof of Theorem. We used the following two Lemmas to prove the Theorem.

Lemma 1. *Assume that u satisfies*

$$\begin{aligned} &\varDelta u(x)=0 && x\in M\backslash\overline{B(\varepsilon\,;w)}\\ &u(x)=0 && x\in\partial M\\ &u(x)=L(\theta) && x=w+\varepsilon(\theta_1,\theta_2),\quad \theta=(\theta_1,\theta_2)\in S^2. \end{aligned}$$

Then,

$$\int_{S^2}\left(\frac{\partial u}{\partial\nu}(x)\right)\Big|_{\partial B_\varepsilon}^2 \varepsilon^2 d\theta\le C(\max_\theta L(\theta)^2+W),$$

where

$$W=(\max L(\theta)^2)^{\sigma/(1+\sigma)}(\|L\|^2_{H^1(S^2)}+\|L\|^2_{C^{(1+\sigma')}(S^2)})^{1/(1+\sigma)}$$

for $\sigma'>\sigma>0$. Here $B_\varepsilon=B(\varepsilon\,;w)$.

Lemma 2 (Osawa [4]). *Assume that* $\mathrm{Ker}\,(\varDelta+\lambda(\varepsilon)pu_\varepsilon^{p-1})=\{0\}$ *for fixed ε. Then,*

$$\lambda'(\varepsilon)=\int_{S^2}(\partial u_\varepsilon/\partial\nu_x)(\varepsilon\theta)^2\varepsilon^2 d\theta,$$

where ν_x denotes the exterior unit normal vector.

The author does not know whether we can remove the assumption in Lemma 2 or not.

The main outline of our proof of the Theorem is in Lemma 3 and (2.1) below.

Let $G_\varepsilon(x,y)$ be the Green function of $-\varDelta$ in M_ε under the Dirichlet condition on ∂M_ε. Let $G(x,y)$ be the Green function of $-\varDelta$ in M under the Dirichlet condition on ∂M.

We introduce the following kernel function $p_\varepsilon(x,y)$. We put

$$p_\varepsilon(x,y)=G(x,y)-4\pi\varepsilon G(x,w)G(w,y).$$

Let $\boldsymbol{G}_\varepsilon$, $\boldsymbol{P}_\varepsilon$ be the operators given by

$$\boldsymbol{G}_\varepsilon f(x)=\int_{M_\varepsilon} G_\varepsilon(x,y)f(y)dy$$

$$\boldsymbol{P}_\varepsilon f(x)=\int_{M_\varepsilon} p_\varepsilon(x,y)f(y)dy.$$

We have the following

Lemma 3. *There exist constants h, $C>0$ such that*

$$\int_{S^2}\left(\left(\frac{\partial}{\partial\nu}(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)\right)u_\varepsilon^p\right)\Big|_{\partial B_\varepsilon}^2 \varepsilon^2 d\theta\le C\varepsilon^h\|u_\varepsilon^p\|_{L^q(M_\varepsilon)}$$

for $q>2$.

We can see that

$$\sup_{\varepsilon} \sup_{x} |u_\varepsilon^p(x)| < C < +\infty$$

holds for $p \in (1,2)$.

We want to calculate the right hand side of

$$\begin{aligned}\lambda(\varepsilon)-\lambda &= \int_0^\varepsilon \lambda'(s)ds \\ &= \int_0^\varepsilon \left(\int_{S^2} (\partial u_t/\partial \nu_x)(t\theta)^2 t^2 d\theta \right) dt.\end{aligned}$$

We have

$$\begin{aligned}(2.1) \qquad \int_{S^2} (\partial u_t/\partial \nu_x)^2 t^2 d\theta = \lambda(t)^2 \Big\{ & \int_{S^2} (\partial \boldsymbol{P}_t u_t^p/\partial \nu_x)^2 t^2 d\theta \\ & +2\int_{S^2} (\partial \boldsymbol{P}_t u_t^p/\partial \nu_x)(\partial(\boldsymbol{P}_t-\boldsymbol{G}_t)u_t^p/\partial \nu_x) t^2 d\theta \\ & +\int_{S^2} (\partial(\boldsymbol{P}_t-\boldsymbol{G}_t)u_t^p/\partial \nu_x)^2 t^2 d\theta \Big\}.\end{aligned}$$

By an explicit calculation of the first term of the right hand side of (2.1) we get the Theorem. Using Lemma 3, we can see that the second term and the third term are negligible.

The details of this paper will appear elsewhere.

References

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