37. On Properties of Non-Carathéodory Functions

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Let N be the class of all functions that are analytic in the unit disk $E=\{z: |z|<1\}$ and equal to 1 at $z=0$. We call $p(z)\in N$ a Carathéodory function, if it satisfies the condition $\operatorname{Re} p(z)>0$ in E.

In this paper, we will obtain some properties of $p(z)\in N$ which is not a Carathéodory function. We need the following lemma due to [1, 2].

Lemma 1. *Let $w(z)$ be analytic in E and suppose that $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then we can write*

$$z_0 w'(z_0)=kw(z_0)$$

where k is a real number and $k\geqq 1$.

Theorem 1. *Let $p(z)\in N$ and suppose that there exists a point $z_0\in E$ such that*

$$\text{(1)}\qquad \begin{aligned} &\operatorname{Re} p(z)>0 \quad \textit{for} \quad |z|<|z_0| \\ &\operatorname{Re} p(z_0)=0 \quad \textit{and} \quad p(z_0)\neq 0. \end{aligned}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)}=ik$$

where k is real and $|k|\geqq 1$.

Proof. Let us put

$$\text{(2)}\qquad \phi(z)=\frac{1-p(z)}{1+p(z)}.$$

Then we have that $\phi(0)=0$, $|\phi(z)|<1$ for $|z|<|z_0|$ and $|\phi(z_0)|=1$.

From (1), (2) and Lemma 1, we easily obtain

$$\frac{z_0\phi'(z_0)}{\phi(z_0)}=\frac{-2z_0p'(z_0)}{1-p(z_0)^2}=\frac{-2z_0p'(z_0)}{1+|p(z_0)|^2}\geqq 1.$$

This shows that

$$-z_0p'(z_0)\geqq\frac{1}{2}(1+|p(z_0)|^2)$$

and $z_0p'(z_0)$ is a negative real number.

Since $p(z_0)$ is a non-vanishing pure imaginary number, we can put

$$p(z_0)=ia,$$

where a is a non-vanishing real number.

For $a>0$, we have

$$\operatorname{Im}\frac{z_0p'(z_0)}{p(z_0)}=\operatorname{Im}\left(-\frac{iz_0p'(z_0)}{|p(z_0)|}\right)$$

$$\geqq \frac{1}{2}\left(\frac{1+a^2}{a}\right) \geqq 1,$$

and for $a<0$, we have

$$\operatorname{Im} \frac{z_0 p'(z_0)}{p(z_0)} = \operatorname{Im} \frac{i z_0 p'(z_0)}{|p(z_0)|}$$
$$\leqq -\frac{1}{2}\left(\frac{1+a^2}{|a|}\right) \leqq -1.$$

This completes our proof.

Theorem 2. *Let $p(z) \in N$ and suppose that*

$$\left|\operatorname{Im} \frac{zp'(z)}{p(z)}\right| < 1 \qquad \text{in } E. \tag{3}$$

Then we have

$$\operatorname{Re} p(z) > 0 \qquad \text{in } E.$$

Proof. From the assumption (3), we easily have

$$p(z) \neq 0 \qquad \text{in } E.$$

In fact, if $p(z)$ has a zero of order n at $z=\alpha$, then we can put

$$p(z) = (z-\alpha)^n p_1(z)$$

where $p_1(z)$ is analytic in E and $p_1(\alpha) \neq 0$. Then we have

$$\frac{zp'(z)}{p(z)} = \frac{nz}{z-\alpha} + \frac{zp_1'(z)}{p_1(z)}. \tag{4}$$

But, the imaginary part of the right-hand side of (4) can take any values when z approaches α. This contradicts (3). This shows that

$$p(z) \neq 0 \qquad \text{in } E.$$

Therefore, if there exists a point $z_0 \in E$ such that

$$\operatorname{Re} p(z_0) = 0,$$

then we have $p(z_0) \neq 0$. From Theorem 1, we obtain

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $k \geqq 1$. This contradicts (3). This completes our proof.

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References

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[2] S. S. Miller and P. T. Mocanu: Second order differential inequalities in the complex plane. J. Math. Anal. Appl., **65**, 289–305 (1978).