36. The Centralizer Algebras of Mixed Tensor Representations of $\mathcal{U}_q(gl_n)$ and the HOMFLY Polynomial of Links[†)]

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Introduction. We construct an algebra $H_{N-1,M-1}(a,q)$ with complex parameters a and q. The centralizer algebra of a mixed tensor representation of $\mathcal{U}_q(gl_n)$ is a quotient of it. The HOMFLY polynomial of links in S^3 is equal to a trace of $H_{N-1,M-1}(a,q)$. Each irreducible character of it corresponds to an invariant of links in a solid torus. As an application, we get a formula for the HOMFLY polynomial of satellite links. The detail will be published elsewhere.

1. The centralizer algebra of mixed tensor representation. The quantum group $\mathcal{U}_q(gl_n)$ is the q-analogue of the universal enveloping algebra $\mathcal{U}(gl_n)$. The Lie algebra gl_n acts on $V_n := \boldsymbol{C}^n$ naturally and it is called the vector representation. This representation can be deformed for the q-analogue $\mathcal{U}_q(gl_n)$ and is also called the *vector* representation. Let V_n^* denote the dual representation of V_n. Since $\mathcal{U}_q(gl_n)$ is a Hopf algebra, it acts on

$$V_n^{(N,M)} := \underbrace{V_n \otimes \cdots \otimes V_n}_{N \text{ times}} \otimes \underbrace{V_n^* \otimes \cdots \otimes V_n^*}_{M \text{ times}}.$$

This representation is called the *mixed tensor* representation of $\mathcal{U}_q(gl_n)$. Let

$$C_n^{(N,M)} := \{x \in \operatorname{End}(V_n^{(N,M)}) \mid xa = ax \text{ for any } a \in \mathcal{U}_q(gl_n)\}.$$

Then $C_n^{(N,M)}$ is an algebra and is called the *centralizer algebra* with respect to $V_n^{(N,M)}$. Jimbo shows in [2] that $C_n^{(N,0)}$ is a quotient of the Iwahori-Hecke algebra $H_{N-1}(q)$. Let q and a be generic complex parameters. In other words, they are not equal to 0 nor any root of unity. Let $H_{N-1,M-1}(a,q)$ be the algebra defined by the following generators and relations.

$$H_{N-1,M-1}(a,q) = \langle T_1^+, \cdots, T_{N-1}^+, T_1^-, \cdots, T_{M-1}^-, E \mid T_i^\pm T_{i+1}^\pm T_i^\pm = T_{i+1}^\pm T_i^\pm T_{i+1}^\pm,$$
$$T_i^\pm T_j^\pm = T_j^\pm T_i^\pm \ (|i-j| \geq 2), \quad T_i^\pm T_j^\mp = T_j^\mp T_i^\pm, \quad ET_i^\pm = T_i^\pm E \ (i \geq 2),$$
$$E(T_1^+)^{-1} T_1^- E T_1^+ = E(T_1^+)^{-1} T_1^- E T_1^-, \quad ET_1^\pm E = a^{-1}E, \quad E^2 = -\frac{a-a^{-1}}{q-q^{-1}}E,$$
$$T_1^+ E (T_1^+)^{-1} T_1^- E = T_1^- E (T_1^+)^{-1} T_1^- E, \quad (T_i^\pm - q)(T_i^\pm + q^{-1}) = 0\rangle.$$

Theorem 1. (1) *The algebra $H_{N-1,M-1}(a,q)$ is semisimple and its dimension is equal to the factorial $(N+M)!$.*

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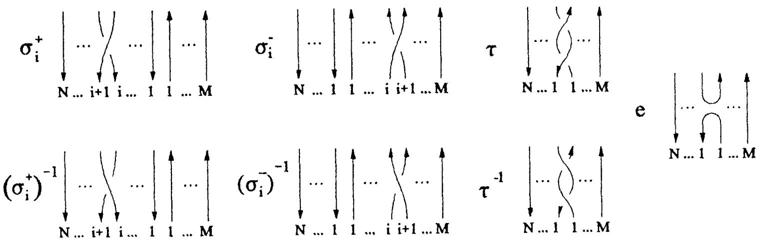
(2) *The centralizer algebra* $C_n^{(N,M)}$ *is isomorphic to a quotient of* $H_{N-1,M-1}(q^{-n},q)$. *If n is sufficiently large, then they are isomorphic.*

In this correspondence, $T_i^{\pm}$ is a scalar multiple of the image of the R-matrix, and E is a scalar multiple of the image of the $\mathcal{U}_q(gl_n)$-module mapping corresponding to the natural pairing $V_n \otimes V_n^* \to \boldsymbol{C}$.

Let Λ_r denote the set of partitions of r. As an abstract algebra, $H_{N-1,M-1}(q^{-n},q)$ is isomorphic to the centralizer algebra of the mixed tensor representation of gl_n if $n \gg 0$. Therefore, the irreducible representations of $H_{N-1,M-1}(a,q)$ are parametrized by the set $\Lambda_{N,M} = \{(\lambda,\mu) \mid \lambda \in \Lambda_{N-k},\ \mu \in \Lambda_{M-k}\ (k \geq 0)\}$ (see, for example, [5]). By using the Bratteli diagram of inclusions $\boldsymbol{C} \subset H_0(q) \subset H_1(q) \subset \cdots \subset H_{N-1}(q) \subset H_{N-1,0}(a,q) \subset \cdots \subset H_{N-1,M-1}(a,q)$, we can actually construct all the irreducible representations of $H_{N-1,M-1}(a,q)$ by the method in [1]. These representations are given in [3] and are generalizations of those of the Iwahori-Hecke algebra in [7].

2. The HOMFLY polynomial. An oriented knit semigroup $B_{N,M}$ is generated by the following elements:



The product of these elements are defined like the braid group. Two elements of $B_{N,M}$ are regarded as the same element if their diagrams are isotopic. By closing an element $b \in B_{N,M}$, we get an oriented link diagram $\hat{b}$ in S^3. Note that this correspondence is well-defined by the definition of the knit semigroup. The HOMFLY polynomial P of links is defined uniquely by the following relation.

$$a^{-1}P_{\times}(a,q) - aP_{\times}(a,q) = (q-q^{-1})P_{\downarrow\downarrow}(a,q), \quad P_{\bigcirc}(a,q) = 1.$$

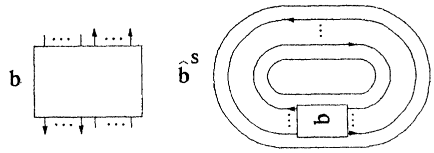
The first relation is called the *skein* relation. Factoring the semigroup algebra $\boldsymbol{C}B_{N,M}$ by the skein relation, we get $H_{N-1,M-1}(a,q)$. The projection map $\iota_{N,M}$ is defined by $\iota_{N,M}(\sigma_i^{\pm}) = aT_i^{\pm}$, $\iota_{N,M}(e^{\pm}) = E$, $\iota_{N,M}(\tau) = -a^{-1}(q-q^{-1})e + a^{-2}$. Let $\chi_{\lambda,\mu}^{(N,M)}$ be the irreducible character of $H_{N-1,M-1}(a,q)$ parametrized by $(\lambda,\mu) \in \Lambda_{N,M}$.

Theorem 2. *For* $b \in B_{N,M}$, *there are* $a_{\lambda,\mu}^{(N,M)} \in \boldsymbol{C}$ *such that* $P(\hat{b}) = \sum_{(\lambda,\mu) \in \Lambda_{N,M}} a_{\lambda,\mu}^{(N,M)} \chi_{\lambda,\mu}^{(N,M)}(\iota_{N,M}(b))$.

The coefficient $a_{\lambda,\mu}^{(N,M)}$ is not depend on b and is given in [3].

3. Invariants of links in a solid torus. Let b be an element of $B_{N,M}$ and let S be a solid torus. Put b in an anulus and close it along with the axis as in the figure, we get a diagram of a link in S, which is denoted by

$\hat{b}^S$. On the other hand, every link in S is realized as a closure of a certain element of $B_{N,M}$. Fix a pair of partitions (λ, μ). Let $\chi_{\lambda,\mu}^{(N,M)}(a, q)$ be the irreducible character of $H_{N-1,M-1}$ parametrized by $(\lambda, \mu) \in \Lambda_{N,M}$, or 0 if there is no $k \geq 0$ such that $|\lambda| = N-k$, $|\mu| = M-k$.



Proposition 3. *Let $b_1 \in B_{N_1,M_1}$ and $b_2 \in B_{N_2,M_2}$. If the closures $\hat{b}_1$ and $\hat{b}_2$ are the equivalent links, then we have $\chi_{\lambda,\mu}^{(N_1,M_1)}(\iota_{N_1,M_1}(b_1)) = \chi_{\lambda,\mu}^{(N_2,M_2)}(\iota_{N_2,M_2}(b_2))$.*

For a link L in the solid torus, let $Q_{\lambda,\mu}(L) = \chi_{\lambda,\mu}^{(N,M)}(\iota_{N,M}(b))$, where $b \in B_{N,M}$ such that $\hat{b}^S$ is equivalent to L. Then the above proposition shows that $Q_{\lambda,\mu}(L)$ is well-defined. In other words,

Corollary 4. *The above formula implies that $Q_{\lambda,\mu}(L)$ is an ambient isotopy invariant of links in the solid torus.*

The idea of the proof is to use the category of tangles in [6].

4. The HOMFLY polynomial of satellite links. A satellite link is a link in S^3 obtained from a knot in S^3 and a link in a solid torus. Let K be a knot in S^3 and let L be a link in a solid torus S. Let $N(K)$ be the tublar neighborhood of K. Then $N(K)$ is isomorphic to the solid torus. Let f be the faithful embedding from the solid torus to $N(K)$. Then the image $f(L)$ is a link in S^3. This link is denoted by K_L and is called a *satellite* of K by L. Let $(\lambda, \mu) \in \Lambda_{N,M}$ and let $\alpha_{\lambda,\mu}$ be the central idempotent of $H_{N-1,M-1}(a, q)$ corresponding to the irreducible representation parametrized by (λ, μ). Then there is $\beta_{\lambda,\mu} \in \boldsymbol{C}B_{N,M}$ such that $\iota_{N,M}(\beta_{\lambda,\mu}) = \alpha_{\lambda,\mu}$, where $\iota_{N,M}$ is the projection from $\boldsymbol{C}B_{N,M}$ to $H_{N,M}(a, q)$. Let $\beta_{\lambda,\mu} = \sum_i c_i \gamma_i$, $(c_i \in \boldsymbol{C}$, $\gamma_i \in B_{N,M})$, and put $P_{\lambda,\mu}^{(N,M)}(K) := \sum_i c_i P(K_{\hat{\gamma}_i^S})$.

Proposition 5. *For $(\lambda, \mu) \in \Lambda_{N,M}$,*

(1) *$P_{\lambda,\mu}^{(N,M)}$ is a knot invariant.*

(2) *$P_{\lambda,\mu}^{(N,M)}(K) = P_{\lambda,\mu}^{(|\lambda|,|\mu|)}(K)$ for any knot K.*

According to (2), we simply denote $P_{\lambda,\mu}^{(N,M)}$ by $P_{\lambda,\mu}$. By using the invariants $P_{\lambda,\mu}$ and $Q_{\lambda,\mu}$ in the last section, we get a formula for the HOMFLY polynomial of satellite links.

Theorem 6. *For any knot K in S^3 and L in a solid torus, we have*

$$P(K_L) = \sum_{(\lambda,\mu)} Q_{\lambda,\mu}(L) P_{\lambda,\mu}(K).$$

Remark. (1) $Q_{\lambda,\mu}(L) = 0$ except with finite number of pairs (λ, μ) of partitions.

(2) For the Jones polynomial and the Kauffman polynomial, such formula is already given in [4].

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