35. *Families of Rational Maps and Convergence Basins of Newton's Method*

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(Communicated by Shokichi IYANAGA, M. J. A., June 9, 1992)

1. Introduction. The subject treated here is an attempt to understand the efficiency of algorithms for solving non-linear equations. Among others, Newton's method plays a central role in root-finding algorithms for polynomials. The global study of this algorithm leads to a theory of complex dynamical systems of rational functions.

We write $N: P_d \times \bar{C} \to \bar{C}$, where P_d is the space of polynomials of degree $\leq d$ and $\bar{C}$ is the Riemann sphere $C \cup \{\infty\}$. Then Newton map, $N(p, z) = N_p(z) = z - p(z)/p'(z)$, is rational over $\bar{C}$ for $p \in P_d$ and $z \in \bar{C}$; that is, N_p can be formed from the complex rational operations from the coefficients of p and z. If z is sufficiently close to a root α of p, then the sequence defined by $z_1 = N_p(z)$, $z_2 = N_p^2(z) = N_p(z_1)$, $\cdots$, $z_k = N_p^k(z) = N_p(z_{k-1})$ converges to α as k tends to ∞. However, as is well known, there is an open set U in $P_d \times \bar{C}$ such that this convergence will not happen for (f, z) in U. Consequently, for Newton's iterative scheme, two distinctly different types of behavior have been observed. In the first case, this algorithm succeeds for an open dense set of starting points. The set of exceptional points is closed, nowhere dense and has two dimensional measure zero. The second case exhibits an open set of initial conditions where this algorithm fails. The failure is due to the existence of an attracting periodic cycle of a Newton map.

In Smale ([10]), he conjectured that there exists no generally convergent purely iterative algorithm for finding roots of polynomials. C. McMullen ([4]) answered the question by showing that there is no such algorithm for polynomials of degree ≥ 4. Here "purely iterative" means that the algorithm can be expressed as a discrete dynamical system on $\bar{C}$ parameterized by the polynomial. However it was shown by M. Shub and S. Smale ([9]) that if one adds the operation of complex conjugation, then there do exist such algorithms.

In this paper, we shall analyze the global behavior of Newton map from the viewpoint of complex dynamics of rational functions. In Section 2, we give a complete criterion for a rational function to be a Newton's method as applied to a polynomial map. In Section 3, we study how one can guarantee success of Newton's method, by measuring the width of basins of roots.

2. Characterization of rational functions to be a Newton map. For a polynomial $p(z)$, we define Newton map as follows.

$$N_p(z)=z-\frac{p(z)}{p'(z)}.$$

Hereafter $p(z)$ is a polynomial of degree d. It is clear that if $p(z)$ has n distinct roots then N_p is a rational function of degree n.

The following facts concerning an immediate basin $B(\alpha)$ ($\overset{\text{def}}{=}\{z\,;\,N_p^n(z)\to\alpha(n\to\infty)\}$) of Newton map N_p are known, where α is a root of p.

1. $B(\alpha)$ is simply connected ([7], [8]).
2. ∞ lies on the boundary of $B(\alpha)$ for every root α of $p(z)$ ([3]).
3. If the local degree of $N_p|_{B(\alpha)}$ is s, then $B(\alpha)$ approaches ∞ in $s-1$ different directions ([7]).

Let x_0 be a periodic point of period s, i.e. $f^s(x_0)=x_0$ for a rational function f. If $x_0\neq\infty$ then we define **eigenvalue** of x_0 as follows: $\lambda=(f^s)'(x_0)$. A periodic point x_0 is said to be **attracting** if $0<|\lambda|<1$, **super-attracting** if $\lambda=0$, **neutral (indifferent)** if $|\lambda|=1$, and **repelling** if $|\lambda|>1$.

A rational function f is (analytic) **conjugate** to a rational function g iff there exists a Möbius transformation $A(z)=(az+b)/(cz+d)\in PSL(2, \boldsymbol{C})$, satisfying $A\circ f(z)=g\circ A(z)$.

The following facts are known on fixed points of N_p.

1. The set of fixed points of $N_p(z)$ is $\{\infty\}\cup p^{(-1)}(0)$.
2. If α is a root of $p(z)$ with multiplicity m, then $N_p'(\alpha)=(m-1)/m$. Hence for $m>1$, α is attractiug and for $m=1$, α is super-attracting.
3. ∞ is the unique repelling fixed point of N_p, and its eigenvalue is $d/(d-1)$.

We have now the following fundamental result.

Theorem 2.1. *The next two statements are equivalent for a rational function f of degree d.*

1. *f has distinct d fixed points, $z_1, z_2, \cdots, z_d$, whose eigenvalues are given as $f'(z_i)=(m_i-1)/m_i$, $m_i\in\boldsymbol{N}$, $i=1, \cdots, d$.*
2. *There exists a polynomial p for which Newton map N_p is conjugate to f.*

This theorem covers the result by J. Head as a corollary:

Corollary 2.2 (Head) ([2]). *Any rational function f of degree d having d distinct super-attracting fixed points is conjugate to the N_p for a polynomial p of degree d.*

For the proof of Theorem 2.1, we need some definitions and results due to Milnor ([5]). The **multiplicity** of a fixed point z_0 of $f(z)$ is defined as follows. If $f'(z_0)\neq1$, multiplicity of z_0 is 1. If $f'(z_0)=1$, then the Taylor expansion of f at z_0 is

$$f(z)=z_0+(z-z_0)+a(z-z_0)^m+\cdots,\quad a\neq0.$$

In this case, multiplicity is defined by m.

The **holomorphic index** ι of a fixed point z_0 is defined as follows:

$$\iota(f\,;\,z_0)=\frac{1}{2\pi i}\int_{|z-z_0|=\delta}\frac{1}{z-f(z)}dz=\operatorname{Res}\left(\frac{1}{z-f(z)}\,;\,z_0\right).$$

Note that the index at z_0 is a local analytic invariant. That is, if g is locally conjugate to f under φ, then $\iota(f\,;z_0)=\iota(g\,;\varphi(z_0))$.

Theorem 2.3 (Milnor) ([5]). *For a rational function* $f(z)$ $(\not\equiv z)$, *we have* $\sum_{f(z)=z}\iota(f\,;z)=1$. *If* $z_0=f(z_0)$, $f'(z_0)=\lambda\neq 1$ *then* $\iota(f\,;z_0)=1/(1-\lambda)$.

Outline of proof of Theorem 2.1. 2. $\Rightarrow$1. is easy. Now we shall show that 1. $\Rightarrow$2. Note that $f(z)$ has precisely $d+1$ fixed points. Let ζ be a fixed point $\neq z_i$ $(i=1,\cdots,d)$. The multiplicity of the fixed point ζ is 1, because that multiplicity of each fixed point z_i $(1\leq i\leq d)$ is 1.

Put $k=\sum_{i=1}^d m_i$. Then we get $f'(\zeta)=k/(k-1)$ from the equation;

$$\sum_{z=f(z)}\iota(f\,;z)=\sum_{i=1}^{d}\frac{1}{1-(m_i-1)/m_i}+\frac{1}{1-f'(\zeta)}=1.$$

Hence it turns out that ζ is a repelling fixed point.

If $\zeta\neq\infty$ then by a change of coordinate, ζ is transformed to ∞, and f to a conjugate rational function $\tilde{f}$. We denote $\tilde{f}$ by the same f for the simplicity of the notation.

Hence we can write $f(z)=q(z)/r(z)$ where $q(z)$ and $r(z)$ are polynomials of $\deg q=d$ and $\deg r<d$.

After calculation, we get $r(z)=\sum_{i=1}^d m_i\cdot\prod_{i\neq j}(z-z_j)$. Put $P_0(z)=c\cdot\prod_{i=1}^d(z-z_j)^{m_i}$. Then we have

$$f(z)=z-\frac{c\cdot\prod_{i=1}^d(z-z_j)}{\sum_{i=1}^d m_i\cdot\prod_{i\neq j}(z-z_j)}\cdot\frac{\prod_{i=1}^d(z-z_i)^{m_i-1}}{\prod_{i=1}^d(z-z_i)^{m_i-1}}$$
$$=z-\frac{P_0(z)}{P_0'(z)}=N_{P_0}(z).$$

3. Sutherland's estimate for the basins of Newton map. In order to have an estimate on the complexity of a root-finding algorithm, we need a compactness condition under a suitable norm on the space of polynomials. This can be done by placing conditions either of the location of the roots or of the coefficients. Hence we consider hereafter a polynomial in the family $\mathcal{P}_d(1)$:

$$\mathcal{P}_d(1)=\{p(z)=z^d+a_{d-1}z^{d-1}+\cdots+a_0,\ |a_i|\leq 1\ (i=0,\cdots,d-1)\}.$$

Moreover if $a_{d-1}=0$ then $p(z)$ is called **centered polynomial.** It is possible to transform linearly an arbitrary polynomial $p(z)$ into an element $q(z)$ of $\mathcal{P}_d(1)$. Note that the Newton map N_p induced by p is conjugate to N_q induced by q. Therefore, without loss of generality, we can treat only centered polynomials $p\in\mathcal{P}_d(1)$, after such conjugacy.

Let $B(\alpha)$ be an immediate basin of attracting fixed point α (a root of p) of N_p. In [10], Smale asks for a lower bound on the area of $B(\alpha)\cap\{z\,;|z|\leq 2\}$. In [3], Manning attempted to estimate the size of $B(\alpha)$, for solving Smale's question. In [11], Sutherland improved Manning's result, but as we have shown in [6], some of his results should be corrected, particularly his Proposition 3.5 which surves as key lemma for his later development. In this section, we shall show that this Proposition 3.5 in [11] can be replaced by the following Theorem 3.2.

Definition 3.1. Any annulus can be mapped by an analytic diffeomorphism onto a unique "standard annulus" whose inner boundary is the unit circle and with outer boundary the circle of radius $e^{2\pi m}$ for some $m \in \boldsymbol{R}^+$. In this case, the modulus of the annulus is said to be m.

Theorem 3.2. *Let $\boldsymbol{T}$ be a torus, isomorphic to $\boldsymbol{C}/(\boldsymbol{Z}\oplus\boldsymbol{Z}\tau)$, and A an annulus with modulus$(A)=m$, contained in T* (see Fig. 1). *Then the distance between the boundary curves of A is at least*

$$\frac{2ke^{m/2\pi}}{1+e^{\pi/m}},$$

where $k=\min\{1, \Im(\tau)\}$,

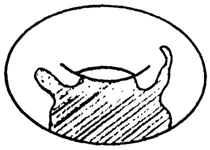


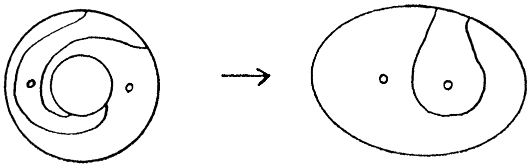
Fig. 1

Proof of Theorem. Consider an open ellipse whose major axis is the interval $\left(-\frac{r+1/r}{2}, \frac{r+1/r}{2}\right)$ and minor axis is $\left(-\frac{r-1/r}{2}i, \frac{r-1/r}{2}i\right)$. Remove two points $-1, 1$ from the ellipse and denote by E the resulting set. Let Γ be the set of curves in E which join the boundary of the ellipse passing through the interval $(-1, 1)$.

The map $z\mapsto(z+1/z)/2$ is two to one. And a punched annulus

$$PA=\left\{z\,;\,\frac{1}{r}<|z|<r\right\}-\{-1,1\}$$

is mapped to E. Let Γ' be the set of curves in PA joining inner boundary and outer boundary.



PA *E*

Fig. 2

Then the extremal lengths are calculated as follows:

$$\lambda(\Gamma')=\frac{1}{\pi}\log r,\quad \text{and}\quad \lambda(\Gamma)=\frac{2}{\pi}\log r.$$

We may assume that the "narrow part" of this embedded annulus is located at the center of T. Let δ be the width of the narrow part of E. Scale the ellipse by $\delta/2$ and embed it in T so that the interval $[-\delta/2, \delta/2]$ corresponds to the narrow part.

Let

$$r=\frac{k}{\delta}+\sqrt{\left(\frac{k}{\delta}\right)^2-1},$$

where $k=\min\{1, \Im(\tau)\}$.

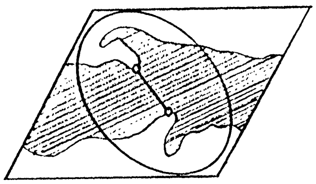


Fig. 3

Let Γ_A be the family of closed curves in the annulus A that have nontrivial homotopy. Then $\lambda(\Gamma_A)=1/m$. Since each curve in Γ_A contains a curves in Γ, we have

$$\frac{1}{m}>\lambda(\Gamma)=\frac{2\log r}{\pi}=\frac{2\log((k/\delta)+\sqrt{(k/\delta)^2-1})}{\pi}.$$

Solving for δ, we obtain

$$\delta\geq\frac{2k\exp(\pi/2m)}{1+\exp(\pi/m)}.$$

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