30. *Closed Regular Curves and the Fundamental Form on the Projective Spaces*

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(Communicated by Heisuke HIRONAKA, M. J. A., June 9, 1992)

0. Introduction. In the previous paper [1] we consider the equivalence classes of closed regular curves in a Riemannian manifold M. The classification is reduced to the problem to solve infinite series of ordinary differential equations whose coefficients are given by curvature tensors of M. The purpose of this paper is to investigate the geometric property of the curvature tensors when the manifold M is either the m-dimensional complex projective space CP_m, the m-dimensional quaternionic projective space HP_m or the Cayley projective plane $\mathfrak{C}P_2$.

In §1 we shall show that the above curvature tensors are written by a local 4-field whose antisymmetrization 4-form Ω_4 is related to the fundamental form on the projective space. Precisely we have the following. For the case $M=CP_m$ and HP_m, the local 4-form Ω_4 defines an invariant 4-form on M. Moreover, if $M=CP_m$, we get $\Omega_4=\Omega_2\wedge\Omega_2$, where Ω_2 is the fundamental form on CP_m. And if $M=HP_m$, the 4-form is coincide with the fundamental form on HP_m defined by Krains [7]. For the case $M=\mathfrak{C}P_2$ we can prove that a local 8-form $\int_{\mathrm{Spin}(9)}\Omega_4\wedge\Omega_4$ gives a fundamental form on $\mathfrak{C}P_2$.

In §2 we shall determine an 8-form on $\mathfrak{C}P_2$ which corresponds to a generator of the integral cohomology ring of $\mathfrak{C}P_2$. For the cases $M=CP_m$ and HP_m, the generator has been determined from the fundamental form, in Besse [4], Chapter 3. The fundamental 8-form on $\mathfrak{C}P_2$ has been obtained by Brown and Gray [5] and Berger [3]. In Besse [4] p. 93, it is asked whether the 8-form furnishes the generator. However it is not easy to do because the 8-form is defined by using integration. We shall find a definitive fundamental 8-form on $\mathfrak{C}P_2$ by the principle of triality, which leads to calculating the generator of the integral cohomology ring.

1. Closed regular curves in the projective spaces. Let M be a projective space CP_m, HP_m or $\mathfrak{C}P_2$. Let $C=\{c(t)\}$ be a closed regular curve in M parameterized by arc length. In [1] we define the equivalence class of C by using a flow on a δ-tubular neighborhood U_δ of C in M. The equation $\psi(t,s)$ of the flow is written as $\psi(t,s)=\exp_{c(t)}sY(t,s)$ $(-\infty<t<\infty,\ -\delta<s<\delta)$, where $Y(t,s)$ is a unit normal vector of C at $c(t)$.

Let G be the isometry group of M and $\{\hat{c}(t)\}$ be a horizontal lift of $\{c(t)\}$ with $\hat{c}(0)=1$. Put $w(t)=(dL_{\hat{c}(t)^{-1}})_{\hat{c}(t)}\hat{c}'(t)$ and $Z(t,s)=(dL_{\hat{c}(t)^{-1}})_{\hat{c}(t)}Y(t,s)$.

Then the equivalence class of C is determined by $Z(t,s)$. We define a 4-field on M as follows. If $M=CP_m$, then $\Omega=\Omega_2\cdot\Omega_2$, where Ω_2 is the fundamental form on CP_m. If $M=HP_m$, then $\Omega=\Omega_I\cdot\Omega_I+\Omega_J\cdot\Omega_J+\Omega_K\cdot\Omega_K$, where Ω_I, Ω_J and Ω_K are 2-forms on the tangent space $T_{c(0)}(M)$ defined by Krains [7], §1. We see that the antisymmetrization 4-form of Ω defines a fundamental 4-form on HP_m (cf. [4], Chapter 3). For the case of $M=\mathfrak{C}P_2$, it is defined by using the associator of the Cayley division algebra $\mathfrak{C}$ (see [2] for details). We can identify $\mathfrak{C}P_2$ with the symmetric space $F_4/\mathrm{Spin}(9)$. The antisymmetrization 4-form Ω_4 is not Spin(9)-invariant. But an 8-form $\int_{\mathrm{Spin}(9)}\Omega_4\wedge\Omega_4$ defines a fundamental 8-form on $\mathfrak{C}P_2$. Let $\{e_i\}$ be a basis of $T_{c(0)}(M)$. From [1], Theorem 3.1 we have the following

Theorem 1.

$$(-\cos 2s+\cos s)\textstyle\sum_i \Omega(Z(t,s),w(t),Z(t,s),e_i)e_i$$
$$+\left(-\frac{1}{2}\sin 2s+\sin s\right)\sum_i\left(\Omega\left(Z(t,s),\frac{\partial Z(t,s)}{\partial t},Z(t,s),e_i\right)\right.$$
$$\left.+\Omega(Z(t,s),w(t),Z(t,s),e_i)\right)e_i+(\sin s)\frac{\partial Z(t,s)}{\partial t}+(\cos s)w(t)$$
$$=\varepsilon(t,s)w(t).$$

Here the summation is taken over $1\le i\le \dim M$ and $\varepsilon(t,s)$ is a real valued function.

2. Generator of the integral cohomology ring *of* $\mathfrak{C}P_2$. In this section we shall determine a generator of the integral cohomology ring of $\mathfrak{C}P_2$. Let T_0 be a tangent space $T_0(\mathfrak{C}P_2)$. Then we can consider T_0 as the ordered pairs (a,b) of Cayley numbers a,b. Let $e_0=1,e_1,\cdots,e_7$ be a basis of $\mathfrak{C}$ as Yokota [8]. Let $v_i,w_i,i=0,\cdots,7$, be 1-forms on T_0 satisfying $v_i(e_j,0)=\delta_{ij}$, $v_i(0,e_j)=0$, $w_i(e_j,0)=0$, $w_i(0,e_j)=\delta_{i,j}$ for $j=0,\cdots,7$. We define a matrix R as follows.

$$R=\begin{pmatrix}0&1&3&2&5&4&7&6\\0&2&1&3&4&6&7&5\\0&3&2&1&7&4&6&5\\0&4&1&5&6&2&3&7\\0&5&4&1&2&7&3&6\\0&6&1&7&2&4&5&3\\0&7&6&1&5&2&4&3\end{pmatrix}.$$

Let J_k $(k=2,3,4)$ be the family of subsets of distinct k elements $(j_1,\cdots,j_k)$ of the set $\{1,2,3,4\}$. Let $R_{i,j}$ denote the (i,j) element of the matrix R. For $1\le i\le 7$, $1\le j\le 4$, put

$$\omega_{ij}=v_{R_{i,2j-1}}\wedge v_{R_{i,2j}},\quad \eta_{ij}=w_{R_{i,2j-1}}\wedge w_{R_{i,2j}}\ (j\ne 1),\ \eta_{i1}=w_i\wedge w_0.$$

We define 8-forms Ω_8^k $(k=1,\cdots,8)$ on T_0 as follows.

$$\Omega_8^1=-14(v_0\wedge\cdots\wedge v_7-w_0\wedge\cdots\wedge w_7).$$

$$\Omega_8^2=-2\sum(\omega_{ij_1}\wedge\omega_{ij_2}\wedge\omega_{ij_3}\wedge\eta_{ij_4}+\eta_{ij_1}\wedge\eta_{ij_2}\wedge\eta_{ij_3}\wedge\omega_{ij_4}),$$

where the summation is taken over $1\le i\le 7$, $(j_1,j_2,j_3,j_4)\in J_4$ with $j_1<j_2<j_3$.

$$\Omega_8^3=-2\sum(-1)^{\varepsilon}(\omega_{ij_1}\wedge\omega_{ij_2}\wedge\omega_{ij_3}\wedge\eta_{ij_1}+\eta_{ij_1}\wedge\eta_{ij_2}\wedge\eta_{ij_3}\wedge\omega_{ij_1}),$$

where the summation is taken over $1\leq i\leq 7$, $(j_1, j_2, j_3)\in J_3$ with $j_2<j_3$ and $\varepsilon=1$ if $j_2=1$ and $\varepsilon=0$ if $j_2>1$.

$$\Omega_8^4=-2\sum \omega_{ij_1}\wedge\omega_{ij_2}\wedge\eta_{ij_3}\wedge\eta_{ij_4},$$

where the summation is taken over $1\leq i\leq 7$, $(j_1, j_2, j_3, j_4)\in J_4$ with $j_1<j_2$ and $j_3<j_4$.

$$\Omega_8^5=2\sum \omega_{ij_1}\wedge\eta_{ij_1}\wedge\omega_{ij_2}\wedge\eta_{ij_2},$$

where the summation is taken over $1\leq i\leq 7$, $(j_1, j_2)\in J_2$ with $j_1<j_2$.

$$\Omega_8^6=-\sum \omega_{i_1j_1}\wedge\eta_{i_1j_2}\wedge\omega_{i_2j_3}\wedge\eta_{i_2j_4},$$

where the summation is taken over $1\leq i_1<i_2\leq 7$, $(j_1, j_2, j_3, j_4)\in J_4$.

$$\Omega_8^7=\sum \omega_{i_1j_1}\wedge\eta_{i_1j_1}\wedge\omega_{i_2j_2}\wedge\eta_{i_2j_2},$$

where the summation is taken over $1\leq i_1<i_2\leq 7$, $(j_1, j_2)\in J_2$.

$$\Omega_8^8=-\sum (-1)^{\varepsilon}\omega_{i_1j_1}\wedge\eta_{i_1j_1}\wedge\omega_{i_2j_2}\wedge\eta_{i_2j_3},$$

where the summation is taken over $1\leq i_1, i_2\leq 7$, $(j_1, j_2, j_3)\in J_3$ and $\varepsilon=1$ if $j_2=1$ or $j_3=1$, otherwise $\varepsilon=0$.

Now we put $\Omega_8=\sum_{i=1}^{8}\Omega_8^i$. Using the principle of triality (see Freudenthal [6]), we have the following

Theorem 2. Ω_8 *is* Spin (9) *invariant.*

By Theorem 2, Ω_8 defines an invariant 8-form on $\mathfrak{C}P_2$. It can be shown that $\Omega_8\wedge\Omega_8=1848\cdot$(volume form of $\mathfrak{C}P_2$). Since the volume of $\mathfrak{C}P_2$ is $8\pi^8/11!$, we have

Theorem 3. $30\sqrt{3}/\pi^4\,\Omega_8$ *gives a generator of integral cohomology ring of* $\mathfrak{C}P_2$.

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