29. On the Universality of Baum-Fulton-MacPherson's Riemann-Roch for Singular Varieties

By Shoji YOKURA
Department of Mathematics, College of Liberal Arts, University of Kagoshima

(Communicated by Kunihiko KODAIRA, M. J. A., June 9, 1992)

§0. Introduction. In [1] Baum, Fulton and MacPherson extended Grothendieck-Riemann-Roch for smooth varieties to singular varieties. Let $\boldsymbol{K}_0$ be the covariant functor of Grothendieck group of coherent algebraic sheaves and $H_{2*}(\ ;\boldsymbol{Q})$ be the even part of the usual $\boldsymbol{Q}$-homology covariant functor. Then they showed the following

BFM-R-R theorem ([1]). *There exists a unique natural transformation* $Td_*: \boldsymbol{K}_0 \to H_{2*}(\ ;\boldsymbol{Q})$ *such that ("smooth Todd-condition")* $Td_*(\mathcal{O}_X) = \mathrm{td}(T_X) \cap [X]$ *for any smooth variety X, where* $\mathcal{O}_X$ *is the structure sheaf of X and* $td(T_X)$ *is the total Todd class of the tangent bundle* T_X *of X.*

The motivation behind establishing this theorem is the following theorem conjectured by Deligne and Grothendiek and proved affirmatively by MacPherson:

DGM-Chern theorem ([4]). *Let* $\mathcal{F}$ *be the constructible function functor. Then there exists a unique natural transformation* $C_*: \mathcal{F} \to H_{2*}(\ ;\boldsymbol{Z})$ *such that ("smooth Chern-condition")* $C_*(\mathbf{1}_X) = c(T_X) \cap [X]$ *for any smooth variety X, where* $\mathbf{1}_X$ *is the charateristic function of X and* $c(T_X)$ *is the total Chern class of the tangent bundle* T_X *of X.*

As to the problem of extending DGM-Chern theorem to other characteristic classes, there is first of all the problem (posed by C. McCrory): *Forgetting "smooth Chern-condition", describe all natural transformations from* $\mathcal{F}$ *to* $H_{2*}(\ ;\boldsymbol{Z})$. To this problem, there is a "naive" conjecture (made by G. Kennedy): *Given a natural transformation* $\tau: \mathcal{F} \to H_{2*}(\ ;\boldsymbol{Z})$, *there exists a sequence* $\{m_i\}_{i\geq 0}$ *of integers* m_i*'s such that* $\tau = \sum_{i\geq 0} m_i C_{*i}$, *where* C_{*i} *is the projection of* C_* *to the 2i-dimensional component.* Kennedy's conjecture is still open, although some partial positive answers have been obtained [6, 8]. In this note we consider Kennedy's conjecture for Baum-Fulton-MacPherson's Riemann-Roch transformation Td_* and show that the conjecture is correct, which becomes another supporting evidence for Kennedy's conjecture.

§1. The universality of BFM's Riemann-Roch transformation Td_*. McCrory's problem seems to be hinted by the classical situation that the total Chern class $c: \boldsymbol{K}^0 \to H^{2*}(\ ;\boldsymbol{Q})^{\wedge} := 1 + \sum_{i\geqq 1} H^{2i}(\ ;\boldsymbol{Q})$ is *universal* for the multiplicative characteristic classes of complex vector bundles, where $\boldsymbol{K}^0$ is the contravariant functor of Grothendieck group of complex vector bundles. To be more precise, for any multiplicative characteristic class $\tau: \boldsymbol{K}^0 \to$

$H^{2*}(\ ;\boldsymbol{Q})^\wedge$ there exists a *unique multiplicative sequence* $\{1, M_1(x_1), M_2(x_1, x_2), \cdots, M_k(x_1, x_2, \cdots, x_k), \cdots\}$ such that $\tau = 1 + \sum_{i\geq 1} M_i(c_1, c_2, \cdots, c_i)$, which means that c is the universal multiplicative characteristic class in the usual sense that the following diagram commutes:

$$\begin{array}{ccc} & \overset{c}{\nearrow} & H^{2*}(\ ;\boldsymbol{Q})^\wedge \\ \boldsymbol{K}^0 & & \big\downarrow \Phi_\tau \\ & \underset{\tau}{\searrow} & H^{2*}(\ ;\boldsymbol{Q})^\wedge \end{array}$$

where Φ_τ is defined by $\Phi_\tau(1+\sum_{i\geq 1} x_i) := 1 + \sum_{i\geq 1} M_i(x_1, x_2, \cdots, x_i)$.

Theorem 1 (*The universality of BFM's Riemann-Roch transformation* Td_*). *For a given natural transformation* $\tau: \boldsymbol{K}_0 \to H_{2*}(\ ;\boldsymbol{Q})$, *there exists a unique sequence* $\{r_i\}_{i\geq 0}$ *of rational numbers* r_i*'s such that* $\tau = \sum_{i\geq 0} r_i Td_{*i}$, *where* Td_{*i} *is the projection of* Td_* *to the* $2i$*-dimensional component. Thus* Td_* *is universal*; $\tau = \Phi_\tau \circ Td_*$, *where* $\Phi_\tau: H_{2*}(\ ;\boldsymbol{Q}) \to H_{2*}(\ ;\boldsymbol{Q})$ *is defined by* $\Phi_\tau(\sum_{i\geq 0} x_i) = \sum_{i\geq 0} r_i x_i$.

The proof of this theorem turns out to be unexpectedly quite simple, unlike in the case of DGM-Chern transformation C_*, thanks to the following strengthened "uniqueness theorem" of BFM-R-R transformation Td_*:

Uniqueness theorem ([1, Chap. III]). *BFM-R-R natural transformation* $Td_*: \boldsymbol{K}_0 \to H_{2*}(\ ;\boldsymbol{Q})$ *is the only natural transformation* τ *satisfying the property that for any projective space* $\boldsymbol{P}^n$ $(n=0,1,2,\cdots)$ *the top-dimensional component of* $\tau(\mathcal{O}_{\boldsymbol{P}^n})$ *is equal to* $[\boldsymbol{P}^n]$, *i.e.,*

$$\tau(\mathcal{O}_{\boldsymbol{P}^n}) = [\boldsymbol{P}^n] + \textit{homology classes of lower degrees.}$$

This theorem follows from the fact that BFM-R-R transformation Td_* induces an isomorphism $\boldsymbol{K}_0(X)\otimes\boldsymbol{Q} \to H_{2*}(X;\boldsymbol{Q})$ for any variety X and the following "identity theorem", the proof of which is attributed to A. Landman:

"Identity theorem" ([2, § 5]). *If* $\alpha: H_{2*}(\ ;\boldsymbol{Q}) \to H_{2*}(\ ;\boldsymbol{Q})$ *is a natural transformation such that for each projective space* $\boldsymbol{P}^i$, $i=0,1,2,\cdots$,

$$\alpha([\boldsymbol{P}^i]) = [\boldsymbol{P}^i] + \textit{homology classes of lower degrees,}$$

then α *must be the identity.*

Proof of Theorem 1: Let $\tau: \boldsymbol{K}_0 \to H_{2*}(\ ;\boldsymbol{Q})$ be a natural transformation and consider all projective spaces $\boldsymbol{P}^i$, $i=0,1,2,\cdots$. Then it is clear that there exists a unique sequence $\{r_i\}_{i\geq 0}$ of rational numbers r_i's such that $\tau(\mathcal{O}_{\boldsymbol{P}^i}) = r_i[\boldsymbol{P}^i]$ + homology classes of lower degrees. Then the "linear" form $\sum_{i\geq 0}(1-r_i)\ Td_{*i}$ is natural and so $\tau' := \tau + \sum_{i\geq 0}(1-r_i)\ Td_{*i}$ is a natural transformation. Then let us "evaluate" this new natural transformation τ' on each projective space $\boldsymbol{P}^i$. By the definition of each rational number r_i above, we have $\tau'(\mathcal{O}_{\boldsymbol{P}^i}) = [\boldsymbol{P}^i]$ + homology classes of lower degrees. Thus by the above BFM's "Uniqueness theorem" we can conclude that $\tau' = Td_*$, which implies that $\tau + \sum_{i\geq 0}(1-r_i)\, Td_{*i} = Td_*$, i.e., $\tau + \sum_{i\geq 0} Td_{*i} - \sum_{i\geq 0} r_i Td_{*i} = Td_*$. Hence, since $Td_* = \sum_{i\geq 0} Td_{*i}$, we get the conclusion that $\tau = \sum_{i\geq 0} r_i Td_{*i}$. Q.E.D.

As corollaries of Theorem 1 and the above proof, we can show the following

Corollary 2 (*A "characterization" of the natural transformation* $\sum_{i\geq 0} r_i Td_{*i}$). *Let* $\{cl^{(n)}\}_{n\geq 0}$ *be a sequence of degree-n characteristic classes* $\mathrm{cl}^{(n)} = \lambda_0^n + \sum_{1\leq i\leq n} P_i^n(c_1, c_2, \cdots, c_i)$, *where each* $P_i^n(c_1, c_2, \cdots, c_i)$ *is a degree-i homogeneous polynomial of individual Chern classes with each* c_k *being of degree k. Then* $\tau: \boldsymbol{K}_0 \to H_{2*}(\ ; \boldsymbol{Q})$ *is a natural transformation satisfying the "dimension-wise universal smooth characteristic condition" that* $\tau(\mathcal{O}_X) = \mathrm{cl}^{(\dim X)}(T_X) \cap [X]$ *for any smooth X if and only if there exists a sequeuce* $\{r_i\}_{i\geq 0}$ *of rational numbers* r_i*'s such that* $\mathrm{cl}^{(n)} = \sum_{0\leq i\leq n} r_i td_{n-i}$ *and* $\tau = \sum_{i\geq 0} r_i Td_{*i}$.

(The proof of this requires also *a generalized "linear independence of Chern numbers"* saying that if cl is a degree-n polynomial of individual Chern classes and $\mathrm{cl}(T_X) = 0$ for any compact smooth variety X of dimension n, then $\mathrm{cl} = 0$ as a polynomial [9].)

Corollary 3. *"Uniqueness theorem" and "Universality" of* Td_* *are equivalent to each other.*

Corollary 4. *Any natural (auto) transformation* $\tau: H_{2*}(\ ; \boldsymbol{Q}) \to H_{2*}(\ ; \boldsymbol{Q})$ *is linear, i.e., given a natural transformation* $\tau: H_{2*}(\ ; \boldsymbol{Q}) \to H_{2*}(\ ; \boldsymbol{Q})$, *there exists a unique sequence* $\{r_i\}_{i\geq 0}$ *of rational numbers* r_i*'s such that* $\tau(\sum_{i\geq 0} x_i) = \sum_{i\geq 0} r_i x_i$. *Namely, if we let* $\pi_i: H_{2*}(\ ; \boldsymbol{Q}) \to H_{2i}(\ ; \boldsymbol{Q})$ *be the projection to the 2i-dimensional component, then* $\tau = \sum_{i\geq 0} r_i \pi_i$.

Corollary 5. *The following three statements are equivalent:*

(i) *Kennedy's conjecture (with* $H_{2*}(\ ; Z)$ *being replaced by* $H_{2*}(\ ; \boldsymbol{Q})$*) is correct.*

(ii) *DGM-Chern transformation* $C_*: \mathcal{F} \to H_{2*}(\ ; \boldsymbol{Q})$ *is universal.*

(iii) *DGM-Chern transformation* $C_*: \mathcal{F} \to H_{2*}(\ ; \boldsymbol{Q})$ *is the only natural transformation* τ *satisfying the property that for each projective space* $\boldsymbol{P}^i$, $i = 0, 1, 2, \cdots$

$$\tau(\mathbf{1}_{P^i}) = [\boldsymbol{P}^i] + \textit{homology classes of lower degrees.}$$

§ 2. An extension of BFM-Riemann-Roch transformation Td_*. In [7] we extended DGM-Chern transformation C_* to the Chern polynomial $c_{(q)} := 1 + \sum_{i\geq 1} q^i c_i$, by introducing the "twisted" constructible function functor $\mathcal{F}^{(q)}$, where $\mathcal{F}^{(q)}(X) := \mathcal{F}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Z}[q]$ and the pushforward $f_*^{(q)}$ for a morphism f involves some kind of twisting. With this twisted constructible function functor $\mathcal{F}^{(q)}$ we could show that there exists a unique natural transformation $C_{(q)*}: \mathcal{F}^{(q)} \to H_{2*}(\ ; Z) \otimes \boldsymbol{Z}[q]$ such that ("smooth Chern polynomial condition") $C_{(q)*}(\mathbf{1}_X) = c_{(q)}(T_X) \cap [X]$ for any smooth X.

In an analogous manner we can extend BFM-Riemann-Roch transformation Td_* to the Todd polynomial $td_{(q)} := 1 + \sum_{i\geq 1} q^i td_i$. Let $\boldsymbol{K}_0^{(q)}(X) := \boldsymbol{K}_0(X) \otimes \boldsymbol{Q}[q, q^{-1}]$ and for a morphism $f: X \to Y$ the pushforward $f_*^{(q)}$ is defined by $f_*^{(q)} := q^{\mathrm{reldim}(f)} f_*$, where $\mathrm{reldim}(f) := \dim X - \dim Y$ and f_* is the usual pushforward. Then it is clear that $\boldsymbol{K}_0^{(q)}$ becomes a covariant functor. Let $H_{2*}^{(q)}(\ ; \boldsymbol{Q}) := H_{2*}(\ ; \boldsymbol{Q}) \otimes \boldsymbol{Q}[q, q^{-1}]$. Then we have the following

Theorem 6 ([10]). *There exists a unique natural transformation* $Td_{(q)*}: \boldsymbol{K}_0^{(q)} \to H_{2*}^{(q)}(\ ;\boldsymbol{Q})$ *such that ("smooth Todd polynomial condition")* $Td_{(q)*}(\mathcal{O}_X) = \mathrm{td}_{(q)}(T_X) \cap [X]$ *for any smooth X. And if we "evaluate"* $Td_{(q)*}$ *at* $q=1$, *then we get the BFM-R-R transformation* $Td_{(1)*} = Td_*$.

References

[1] P. Baum, W. Fulton and R. MacPherson: Riemann-Roch for singular varieties. Publ. Math. I.H.E.S., **45**, 101–145 (1975).
[2] W. Fulton: Rational equivalence on singular varieties. ibid., **45**, 147–167 (1975).
[3] F. Hirzebruch: Topological Methods in Algebraic Geometry. 3rd ed., Springer-Verlag (1966).
[4] R. MacPherson: Chern classes for singular algebraic varieties. Ann. of Math., **100**, 423–432 (1974).
[5] J. M. Milnor and J. D. Stasheff: Characteristic Classes. Ann. of Math. Studies, no. 76, Princeton Univ. Press (1974).
[6] S. Yokura: Some variants of Deligne-Grothendieck-MacPherson's natural transformation C_* of Chern class. Crelles Journal, **419**, 199–211 (1991).
[7] ——: An extension of Deligne-Grothendieck-MacPherson's natural transformation C_* of Chern classes for singular algebraic varieties. Publ. RIMS. Kyoto Univ., **27**, no. 5, 745–762 (1991).
[8] ——: On a generalization of MacPherson's Chern homology class. III. Proc. Japan Acad., **67A**, 260–262 (1991).
[9] ——: A note on linear independence of Chern numbers and Pontryagin numbers. Math. Japon, **37**, no. 6 (1992).
[10] ——: An extension of Baum-Fulton-MacPherson's Riemann-Roch for singular varieties (in preparation).