3. On the Inseparable Degree of the Gauss Map of Higher Order for Space Curves

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Abstract: Let X be a curve non-degenerate in a projective space $\boldsymbol{P}^N$ defined over an algebraically closed field of positive characteristic p, consider the Gauss map of order m defined by the osculating m-planes at general points of X, and denote by $\{b\}_{0\leq j\leq N}$ the orders of X. We prove that the inseparable degree of the Gauss map of order m is equal to the highest power of p dividing b_{m+1}.

Key words: Space curve, Gauss map, inseparable degree.

0. Introduction. Let X be an irreducible curve in a projective space $\boldsymbol{P}^N$ defined over an algebraically closed field k of positive characteristic p, C the normalization of X, and $\iota: C\to\boldsymbol{P}^N$ the natural morphism. Denote by $\iota^{(m)}: C\to\boldsymbol{G}(\boldsymbol{P}^N, m)$ the Gauss map of order m defined by the osculating m-planes of X, where $\boldsymbol{G}(\boldsymbol{P}^N, m)$ is a Grassmann manifold of m-planes in $\boldsymbol{P}^N$. Assume that X is non-degenerate in $\boldsymbol{P}^N$, and let $\{b_j\}_{0\leq j\leq N}$ be the orders of ι. The purpose of this short note is to prove

Theorem. *The inseparable degree of $\iota^{(m)}$ is the highest power of p dividing b_{m+1}.*

In case of $m=1$, Theorem is known: For $N=2$, see [4, Proposition 4.4]; for a general N, see [5, Remark below Corollary 2.3], [3, Proposition 4]. A corollary to this result will give a generalization of [5, Theorem 2.1] (see Corollary below).

In case of $m=N-1$, Theorem coincides with a result of A. Hefez and N. Kakuta, announced in [1]. Although it has not been published yet, according to Hefez [2], their proof for the theorem is similar in spirit to ours (precisely speaking, of the first version), but not identical. Hefez and Kakuta moreover found

Theorem (Hefez-Kakuta). *Denote by $C^{(m)}X$ the conormal variety of order m, and by $X^{*(m)}$ the m-dual. Then the inseparable degree of the natural morphism $C^{(m)}X\to X^{*(m)}$ is equal to the highest power of p dividing b_{m+1}.*

This result is stated as a theorem in [2] without proof.

We finally mention that this Theorem of Hefez and Kakuta is deduced also from our theorem and a result in [6] that $C^{(m)}X\to X^{*(m)}$ *has the same inseparable degree as* $\iota^{(m)}$, which is proved directly without going through

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the orders.

1. Notations and terminology. Denote by V the image of the natural map $\iota^*: H^0(\boldsymbol{P}^N, \mathcal{O}_{\boldsymbol{P}^N}(1)) \to H^0(C, \iota^*\mathcal{O}_{\boldsymbol{P}^N}(1))$. For a general point P in C, choose a trivialization $\iota^*\mathcal{O}_{\boldsymbol{P}^N}(1)_{,P} \simeq \mathcal{O}_{C,P}$, and consider V as a subset of the rational function field $K(C)$. Then, taking a suitable basis $\{x_0, \cdots, x_N\}$ for V, one can expand x_i as follows:

$$(*) \qquad x_i = \sum_{n \geq b_i} a_{i,n} t^n$$

in $\hat{\mathcal{O}}_{C,P} \simeq k[\![t]\!]$ with $a_{i,b_i} \neq 0$, for some non-negative integers $b_0 < b_1 < \cdots < b_N$, where t is a local parameter of C at P. The *orders* of ι are defined to be $\{b_j\}_{0 \leq j \leq N}$ and the *osculating m-plane* of ι at P, denoted by $T_P^{(m)}$, is a linear space determined by $X_{m+1} = \cdots = X_N = 0$, where $(X_0 : \cdots : X_N)$ is a homogeneous coordinate of $\boldsymbol{P}^N$ corresponding to the basis $\{x_i\}_{0 \leq i \leq N}$. The *Gauss map of order m* of ι, denoted by $\iota^{(m)}$, is defined as the elimination of the base locus of the rational map $C \to \boldsymbol{G}(\boldsymbol{P}^N, m)$, sending P to $T_P^{(m)}$ for general points P of C, which is locally given by a matrix, $(D_t^{(b_j)} x_i)_{0 \leq i \leq N, 0 \leq j \leq m}$, where $D_t^{(k)}$ is the Hasse differential operator of order k with respect to the parameter t.

For full details of the above, we refer to [7, §§4, 5], [9, §1].

2. Proof of Theorem. Let q be the highest power of p dividing b_{m+1}, and denote by c the quotient:

$$b_{m+1} = cq.$$

Since $(c-1)q$ is non-negative and p-adically less than b_{m+1}, according to [8, Satz 6], it is an order of ι, say b_k, with $0 \leq k \leq m$.

Lemma. (a) *$D_t^{(b_j)} x_i$ is a unit if $j = i$, and is not if $j < i$.*

(b) *$D_t^{(b_j)} x_i$ is of order at least q if $j \leq m < i$.*

(c) *$D_t^{(b_k)} x_{m+1}$ is of order exactly q.*

Proof. We first note that

$$(*) \qquad D_t^{(b_j)} x_i = \sum_{n \geq b_i} a_{i,n} \binom{n}{b_j} t^{n-b_j}$$

for arbitrary i, j. Thus, (a) follows from this (*).

For (b), suppose the contrary: According to (*), there would exist an integer $n \geq b_i$ such that

$$(**) \qquad \binom{n}{b_j} \not\equiv 0 \mod p, \quad \text{and}$$

$$(***) \qquad n - b_j < q.$$

Dividing by q, writing n and b_j as $n = n'q + n''$, $b_j = b'q + b''$ with $0 \leq n''$, $b'' < q$, we have

$$\binom{n}{b_j} = \binom{n'q + n''}{b'q + b''} \equiv \binom{n'}{b'}\binom{n''}{b''} \mod p.$$

In particular, $n'' \geq b''$ because of (**). On the other hand, $n' \geq c > b'$ since $n \geq b_i \geq b_{m+1} > b_j$ and $b_{m+1} = cq$. Therefore, it follows

$$n - b_j = (n' - b')q + (n'' - b'') \geq q,$$

which contradicts to (***).

Finally for (c), the first term of the summation in (*) for $D_t^{(b_k)}x_{m+1}$ is

$$\binom{b_{m+1}}{b_k}t^{b_{m+1}-b_k},$$

and this is not zero: In fact, we have $\binom{b_{m+1}}{b_k}\equiv\binom{c}{c-1}=c\not\equiv 0 \bmod p$; because $b_{m+1}=cq$, $b_k=(c-1)q$, and c is coprime to p. This proves (c) since $b_{m+1}-b_k=q$.

Now we prove Theorem. Let $\Delta_{i_0,\cdots,i_m}$ be the submatrix of $(D_t^{(b_j)}x_i)_{0\leq i\leq N,\, 0\leq j\leq m}$, consisting of the rows $i_0,\cdots,i_m$, where we start with zero to count those rows. Since the rational map sending P to $T_P^{(m)}$ is determined by the minors,

$$(\det \Delta_{i_0,\cdots,i_m})_{0\leq i_0<\cdots<i_m\leq N}$$

via the Plücker embedding of $\boldsymbol{G}(\boldsymbol{P}^N, m)$, it suffices to show:

(1) $\det \Delta_{0,\cdots,m}$ *is a unit*;

(2) $\det \Delta_{i_0,\cdots,i_m}$ *is of order at least* q *unless* $(i_0,\cdots,i_m)=(0,\cdots,m)$;

(3) $\det \Delta_{0,1,\cdots,k-1,k+1,\cdots,m+1}$ *is of order exactly* q.

Proof. (1) It follows from Lemma (a) that $\Delta_{0,\cdots,m}$ mod t is lower triangular and each entry on the diagonal is a unit. So $\det \Delta_{0,\cdots,m}$ is also a unit as required.

(2) If $(i_0,\cdots,i_m)\neq(0,\cdots,m)$, then $i_m>m$. It follows from Lemma (b) that each entry in the last row $\Delta_{i_0,\cdots,i_m}$ has order at least q, which implies (2).

(3) Consider the matrix

$$\Delta_{0,\cdots,k-1,m+1,k+1,\cdots,m}$$

instead of $\Delta_{0,\cdots,k-1,k+1,\cdots,m+1}$. Ignoring the k-th row, by the similar way to (1) we see that this is equal to a lower triangular matrix mod t and the entries on the diagonal are all units. For the k-th row, according to (b) and (c) in Lemma, every entry has order at least q and the one on the diagonal has order exactly q. We thus conclude that its determinant is of order exactly q, and this completes the proof of (3).

We have thus proved Theorem.

Corollary (Generic Projection). *The inseparable degree of* $\iota^{(m)}$ *is invariant under a projection of* $\boldsymbol{P}^N$ *from a generic point, where* $1\leq m\leq N-2$.

Proof. Since the order b_{m+1} is invariant under a generic projection, the result thus follows from Theorem.

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Addendum. After the draft was written up, we received a preprint "On the geometry of non-classical curves" by A. Hefez and N. Kakuta, which relates to the work mentioned in Introduction.

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