1. *On the Poincaré-Bogovski Lemma on Differential Forms*

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1. Introduction. Integrability conditions for differential forms go back to Poincaré [10, Section II]. Let D be a bounded domain in $\boldsymbol{R}^n$. What is called the Poincaré lemma (cf. [11, Theorem 4.11]) asserts that every smooth closed differential form on D is exact provided that D is starshaped. This is proved by constructing a (linear) integral operator I such that

(1) $$d(I\omega)+I(d\omega)=\omega,$$

where ω is a form on D and d denotes the exterior derivative. Indeed, $d\omega=0$ implies that ω has a potential $I\omega$. However, for usual choice of I, found for example in [11, Theorem 4-11], the support of $I\omega$, $\operatorname{spt} I\omega$, may not be compact in D even if ω is compactly supported in D.

Our goal in this paper is to construct an integral operator K satisfying (1) with $I=K$ such that $\operatorname{spt} K\omega$ is compact if $\operatorname{spt}\omega$ is compact. (More precisely we will show that $\operatorname{spt} K\omega\subset D\cup\Gamma$ if $\operatorname{spt}\omega\subset D\cup\Gamma$ where Γ is an open subset on ∂D.) We also prove that K is bounded in L^p Sobolev spaces.

Bogovski [1], [2] first constructed such K on n-forms ω satisfying $\int_D \omega=0$ (even for an arbitrary bounded Lipschitz domain D); in this case d equals the divergence operator. As noticed in [1, Theorem 4] such a property on $K\omega$ is important for localizing a closed form by preserving closedness. His operator K is applied to various analyses on incompressible viscous fluid (cf. [3], [4], [6], [7], [9], [12], [13]).

Borchers and Sohr [5] and Griesinger [8] treated such a problem on the operator rot. In fact Griesinger [8] constructed an integral operator on a bounded domain D starshaped with respect to a ball in D although she didn't prove (1).

In this paper we extend Bogovski's formula for the exterior derivatives on a bounded domain starshaped with respect to a ball.

2. Formula of potentials. We first give an explicit formula of K. Let $D\subset\boldsymbol{R}^n$ be a bounded domain starshaped with respect to a closed ball B in D, i.e., $D=\{tx+(1-t)y \mid x\in D,\ y\in B,\ t\in[0,1]\}$. Let B' be a closed ball in the interior of B. For $k=1,\cdots,n$ and given $h\in C^\infty(B)$ satisfying $\operatorname{spt} h\subset B'$ and $\int_{B'} h\,dx=1$, we set

$$H_k(x,y)=\int_1^\infty h(y+t(x-y))t^{k-1}(t-1)^{n-k}dt.$$

Let $\mathcal{D}^k$ denote the space of C^∞ k-forms compactly supported in D. For

$\omega \in \mathcal{D}^k$ we recall the exterior derivative $d\omega \in \mathcal{D}^{k+1}$ of ω ;

$$d\omega = \sum_{i_1<\cdots<i_k} \sum_{j\neq i_1,\cdots,i_k} \frac{\partial}{\partial x^j} f_{i_1\cdots i_k} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where

$$\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We define $K_k\omega \in \mathcal{D}^{k-1}$ by

$$K_k\omega(x) = \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \int_D (x-y)^{i_\alpha} H_k(x,y) f_{i_1\cdots i_k}(y) dy \; dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_k}$$

(the symbol $\wedge$ over dx^{i_α} indicates that it is omitted). Since the integral kernel of K_k is integrable, K_k can be extended to a bounded linear operator on L_p^k, where L_p^k denotes the space of p-th integrable k-forms on D. We denote the convex hull spanned by sets A and B by $[A ; B] = \{tx+(1-t)y \mid x \in A, y \in B, t \in [0,1]\}$ and the diameter of A by diam A.

Remark. Bogovski [1] constructed $H_n(x, y)$ as a potential of the operator div and Griesinger [8] constructed $H_2(x, y)$ as a potential of the operator rot.

Theorem. (i) *Assume that* $1 \leq p < \infty$.

(a) *For any* $\omega \in \mathcal{D}^k$, $\operatorname{spt} K_k\omega \subset [\operatorname{spt}\omega ; B']$.

(b) *Suppose that* Γ *is an open subset on* ∂D. *Then for any* $\omega \in L_p^k$, $\operatorname{spt}\omega \subset D \cup \Gamma$ *implies* $\operatorname{spt} K_k\omega \subset D \cup \Gamma$.

(ii) (a) *For* $k=1, \cdots, n-1$, *it holds that*

$$d(K_k\omega) + K_{k+1}(d\omega) = \omega \qquad \text{for all } \omega \in \mathcal{D}^k.$$

(b) *For* $k=n$, *it holds that*

$$d(K_n\omega) = \omega \qquad \text{for all } \omega \in \mathcal{D}^n \text{ with } \int_D \omega = 0.$$

(iii) *Let* $m=0,1,2,\cdots$ *and* $p \in (1,\infty)$. *Then it holds that*

$$\|\nabla^{m+1} K_k\omega\|_p \leq C \|\nabla^m \omega\|_p \qquad \text{for all } \omega \in \mathcal{D}^k$$

with $C = C(n, k, m, p, \operatorname{diam} D, B')$. *Here* $\|\cdot\|_p$ *denotes the* L_p*-norm on* D *and* $\nabla^m f$ *denotes the tensor consisting of all* m*-th derivatives of coefficients of* f.

Remark. The estimate (iii) shows that (ii) holds for all $\omega \in L_p^k$.

3. Proofs. Since (ii)(b) and (iii) can be proved in a similar way to [5, Theorem 2.4], we here only prove (i) and (ii)(a).

(i)(a) By the definition of $K_k\omega$, $x \in \operatorname{spt} K_k\omega$ implies $y+t(x-y) \in B'$ for some $t \geq 1$ and $y \in \operatorname{spt}\omega$. On the other hand for any $x \in D$, $y \in \operatorname{spt}\omega$ and $t \geq 1$, $y+t(x-y) \in B'$ implies $x \in [\operatorname{spt}\omega ; B']$ since $x = t^{-1}(y+t(x-y)) + (1-t^{-1})y$.

(i)(b) For $\delta > 0$ let U_δ be an open set given by $U_\delta = \{x \in D \mid \operatorname{dist}(x, \operatorname{spt}\omega) < \delta\}$. There exist $\omega_j \in \mathcal{D}^k$ such that $\operatorname{spt}\omega_j \subset U_\delta$ and $\omega_j \to \omega$ in L_p^k. Since $[\operatorname{spt}\omega_j ; B'] \subset [U_\delta ; B']$, (i)(a) yields $\operatorname{spt} K_k\omega_j \subset [U_\delta ; B']$. We can see $\overline{[U_\delta ; B']} \cap \partial D = \overline{U_\delta} \cap \partial D$ (see [12, Lemma 3.2]). Since $\delta > 0$ is arbitrary and Γ is open, we obtain (i)(b).

(ii)(a) For simplicity we write $\widehat{\{dx^{i_\alpha}\}}:=dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_\alpha}}\wedge\cdots\wedge dx^{i_k}$ and $d\omega=\sum_{i_1<\cdots<i_k} df_{i_1\cdots i_k}$, where

$$df_{i_1\cdots i_k}=\sum_{j\neq i_1,\cdots,i_k}\frac{\partial}{\partial x^j}f_{i_1\cdots i_k}dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

and $K_k\omega=\sum_{i_1<\cdots<i_k}K_kf_{i_1\cdots i_k}$ in the same way. For $\varepsilon>0$ we set a truncated integration by

$$K_k^\varepsilon f_{i_1\cdots i_k}(x)=\sum_{\alpha=1}^k(-1)^{\alpha-1}\int_{D_\varepsilon}(x-y)^{i_\alpha}H_k(x,y)f_{i_1\cdots i_k}(y)dy\widehat{\{dx^{i_\alpha}\}}$$

where $D_\varepsilon=\{y\in D;|x-y|\geq\varepsilon\}$. Our goal is to prove that the operator $\mathcal{T}_\varepsilon: \mathcal{C}\to\mathcal{C}$ defined by

$$\mathcal{T}_\varepsilon f_{i_1\cdots i_k}=d(K_k^\varepsilon f_{i_1\cdots i_k})+K_{k+1}^\varepsilon(df_{i_1\cdots i_k})$$

converges to the identity operator in the strong topology, namely that $\mathcal{T}_\varepsilon\omega\to\omega$ in $\mathcal{C}$ for all $\omega\in\mathcal{D}^k$, where $\mathcal{C}$ is the space of continuous k-forms on $\bar{D}$. In what follows we consider each component $f_{i_1\cdots i_k}$ so we suppress its subscript. Since (i)(a) implies $K_k\omega(x)=0$ on ∂D, applying the chain rule yields

$$\begin{aligned}d(K_k^\varepsilon f)=\sum_{\alpha=1}^k(-1)^{\alpha-1}\sum_{j\neq\widehat{\{i_\alpha\}}}&\Bigg[\int_{D_\varepsilon}\frac{\partial}{\partial x^j}\{(x-y)^{i_\alpha}H_k(x,y)\}f(y)dy\\&+\int_{|x-y|=\varepsilon}(x-y)^{i_\alpha}H_k(x,y)f(y)\frac{(x-y)^j}{|x-y|}d\sigma_y\Bigg]dx^j\wedge\widehat{\{dx^{i_\alpha}\}}\\=V_1+S_1.\end{aligned}$$

Here $\widehat{\{i_\alpha\}}:=i_1,\cdots,i_{\alpha-1},i_{\alpha+1},\cdots,i_k$ and σ_y denotes the areal element of the sphere $|x-y|=\varepsilon$. On the other hand, we obtain via integrating by parts,

$$\begin{aligned}K_{k+1}^\varepsilon(df)=&\int_{D_\varepsilon}(x-y)^jH_{k+1}(x,y)\sum_{j\neq i_1,\cdots,i_k}\frac{\partial}{\partial y^j}f(y)dy\widehat{dx^j}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\&-\sum_{\alpha=1}^k(-1)^{\alpha-1}\int_{D_\varepsilon}(x-y)^{i_\alpha}H_{k+1}(x,y)\sum_{j\neq i_1,\cdots,i_k}\frac{\partial}{\partial y^j}f(y)dydx^j\wedge\widehat{\{dx^{i_\alpha}\}}\\=&\Bigg[-\sum_{j\neq i_1,\cdots,i_k}\int_{D_\varepsilon}\frac{\partial}{\partial y^j}\{(x-y)^jH_{k+1}(x,y)\}f(y)dydx^{i_1}\wedge\cdots\wedge dx^{i_k}\\&+\sum_{\alpha=1}^k\sum_{j\neq i_1,\cdots,i_k}(-1)^{\alpha-1}\int_{D_\varepsilon}\frac{\partial}{\partial y^j}\{(x-y)^{i_\alpha}H_{k+1}(x,y)\}f(y)dydx^j\wedge\widehat{\{dx^{i_\alpha}\}}\Bigg]\\&+\Bigg[\sum_{j\neq i_1,\cdots,i_k}\int_{|x-y|=\varepsilon}(x-y)^jH_{k+1}(x,y)f(y)\frac{(x-y)^j}{|x-y|}d\sigma_ydx^{i_1}\wedge\cdots\cdot\wedge dx^{i_k}\\&-\sum_{\alpha=1}^k\sum_{j\neq i_1,\cdots,i_k}(-1)^{\alpha-1}\int_{|x-y|=\varepsilon}(x-y)^{i_\alpha}H_{k+1}(x,y)f(y)\frac{(x-y)^j}{|x-y|}d\sigma_ydx^j\wedge\widehat{\{dx^{i_\alpha}\}}\Bigg]\\=&V_2+S_2.\end{aligned}$$

It remains to prove that $V_1+V_2=0$ and $S_1+S_2\to f$ in $\mathcal{C}$ as $\varepsilon\downarrow 0$.

The Leibniz rule yields

$$\begin{aligned}V_1=k&\int H_k(x,y)f(y)dydx^{i_1}\wedge\cdots\wedge dx^{i_k}\\&+\sum_{\alpha=1}^k(-1)^{\alpha-1}\sum_{j\neq\widehat{\{i_\alpha\}}}\int(x-y)^{i_\alpha}\frac{\partial}{\partial x^j}H_k(x,y)f(y)dydx^j\wedge\widehat{\{dx^{i_\alpha}\}},\end{aligned}$$

$$V_2=\int\Big\{(n-k)H_{k+1}(x,y)f(y) - \sum_{j\neq i_1,\cdots,i_k}(x-y)^j\Big(\frac{\partial}{\partial y^j}H_{k+1}(x,y)\Big)f(y)\Big\}dydx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$+\sum_{\alpha=1}^{k}\sum_{j\neq i_1,\cdots,i_k}(-1)^{\alpha-1}\int(x-y)^{i_\alpha}\Big(\frac{\partial}{\partial y^j}H_{k+1}(x,y)\Big)f(y)dydx^j\wedge\widehat{\{dx^{i_\alpha}\}}.$$

Here and hereafter the domain D_ε of volume integrations is suppressed. Noting that

$$\frac{\partial}{\partial y^j}H_{k+1}(x,y)=-\frac{\partial}{\partial x^j}H_k(x,y),$$

we calculate V_1+V_2 by using trivial identities

$$\sum_{j\neq\{\widehat{i_\alpha}\}}a^j-\sum_{j\neq i_1,\cdots,i_k}a^j=a^{i_\alpha} \tag{2}$$
$$(-1)^{\alpha-1}dx^{i_\alpha}\wedge\widehat{\{dx^{i_\alpha}\}}=dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

and obtain

$$V_1+V_2=\int\{kH_k(x,y)+(n-k)H_{k+1}(x-y)\}f(y)dydx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$+\sum_{\alpha=1}^{k}(-1)^{\alpha-1}\int(x-y)^{i_\alpha}\Big(\frac{\partial}{\partial x^{i_\alpha}}H_k(x,y)\Big)f(y)dydx^{i_\alpha}\wedge\widehat{\{dx^{i_\alpha}\}}$$
$$+\sum_{j\neq i_1,\cdots,i_k}\int(x-y)^j\Big(\frac{\partial}{\partial x^j}H_k(x,y)\Big)f(y)dydx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$=\int\Big\{kH_k(x,y)+(n-k)H_{k+1}(x,y) + \sum_{j=1}^{n}(x-y)^j\frac{\partial}{\partial x^j}H_k(x,y)\Big\}f(y)dydx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$=\int\Big[\int_1^\infty\frac{\partial}{\partial t}\{h(y+t(x-y))t^k(t-1)^{n-k}\}dt\Big]f(y)dydx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$=0.$$

We next show that $\lim_{\varepsilon\downarrow 0}(S_1+S_2)=f$ in $\mathcal{C}$. Applying transformations $t=\tau/|x-y|$ and $\tau=s+|x-y|$ to $H_k(x,y)$ yields

$$H_k(x,y)=\frac{1}{|x-y|^n}\int_0^\infty h\Big(x+s\frac{x-y}{|x-y|}\Big)(s+|x-y|)^{k-1}s^{n-k}ds.$$

Since $\mathrm{dist}(x,x+s(x-y)/|x-y|)=s$, $x+s(x-y)/|x-y|\notin\mathrm{spt}\,h$ for any $x,y\in D$ if $s\geq l:=\mathrm{diam}\,D$. Through the binomial expansion $H_k(x,y)$ is now rewritten as follows;

$$H_k(x,y)=\sum_{\beta=0}^{k-1}\binom{k-1}{\beta}G_\beta(x,y),$$

where

$$G_\beta(x,y):=\frac{1}{|x-y|^{n-\beta}}\int_0^l h\Big(x+s\frac{x-y}{|x-y|}\Big)s^{n-1-\beta}ds.$$

This expression implies

$$S_1=\sum_{\alpha=1}^{k}(-1)^{\alpha-1}\sum_{j\neq\{\widehat{i_\alpha}\}}\int_{|x-y|=\varepsilon}\frac{(x-y)^{i_\alpha}(x-y)^j}{|x-y|}$$
$$\times\sum_{\beta=0}^{k-1}\binom{k-1}{\beta}G_\beta(x,y)f(y)d\sigma_y dx^j\wedge\widehat{\{dx^{i_\alpha}\}},$$

$$S_2=\sum_{j\neq i_1,\cdots,i_k}\int_{|x-y|=\varepsilon}\frac{(x^j-y^j)^2}{|x-y|}f(y)\sum_{\beta=0}^{k}\binom{k}{\beta}G_\beta(x,y)d\sigma_y dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$-\sum_{\alpha=1}^{k}\sum_{j\neq i_1,\cdots,i_k}(-1)^{\alpha-1}\int_{|x-y|=\varepsilon}\frac{(x-y)^{i_\alpha}(x-y)^j}{|x-y|}f(y)$$
$$\times\sum_{\beta=0}^{k}\binom{k}{\beta}G_\beta(x,y)d\sigma_y dx^j\wedge\widehat{\{dx^{i_\alpha}\}}.$$

We simply denote

$$S_1+S_2=\int_{|x-y|=\varepsilon}\Big(\sum_{\beta=0}^{k}A_\beta(x,y)\Big)f(y)d\sigma_y.$$

For $\beta=0$, applying (2) to S_1 and the second term in S_2 yields

$$A_0=|x-y|G_0(x,y)dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

The other terms can be ignored. Indeed, letting $\varepsilon\to 0$ yields

$$\sup_{x\in D}\int_{|x-y|=\varepsilon}\Big|\sum_{\beta=1}^{k}A_\beta(x,y)f(y)\Big|d\sigma_y\to 0$$

through estimates

$$\Big|\sum_{\beta=1}^{k}A_\beta(x,y)\Big|\leq C(n,k)\sum_{\beta=1}^{k}|x-y||G_\beta(x,y)|,$$
$$|G_\beta(x,y)|\leq|x-y|^{-n+\beta}\|h\|_\infty\int_0^l s^{n-1-\beta}ds.$$

Let $T_\varepsilon: C\to C$ be the operators defined by

$$T_\varepsilon f(x):=\int_{|x-y|=\varepsilon}A_0(x,y)f(y)d\sigma_y$$
$$=\int_{|z|=1}\Big(\int_0^l h(x+sz)s^{n-1}ds\Big)f(x-\varepsilon z)d\sigma_z dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

(via transformation $x-y=\varepsilon z$). The operators T_ε on C are bounded and $\{T_\varepsilon f\}$ is a Cauchy sequence in C in ε as ε tends to zero for all $f\in C$. There thus exists the limit operator T, which is given by $Tf=\lim_{\varepsilon\downarrow 0}T_\varepsilon f$. We obtain

$$Tf(x)=\Big\{\int_{|z|=1}\Big(\int_0^l h(x+sz)s^{n-1}ds\Big)d\sigma_z\Big\}f(x)$$
$$=\Big(\int_D h(y)dy\Big)f(x)=f(x).$$

4. Remark. Our potential $K_k\omega$ is considered as a variant of usual potential in the Poincaré lemma. Indeed, let $h=h_R\in C_0^\infty(B_R)$ be supported in B_R such that h_R converges to the δ-function as $R\to 0$, where B_R is the ball centered at 0 with radius R. Then $K_k\omega$ converges to

$$J_k\omega(x)=(-1)^{n+1}\sum_{i_1<\cdots<i_k}\sum_{\alpha=1}^{k}(-1)^{\alpha-1}\Big(\int_1^\infty s^{k-1}f_{i_1\cdots i_k}(sx)ds\Big)x^{i_\alpha}$$
$$dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_\alpha}}\wedge\cdots\wedge dx^{i_k}.$$

Note that this is a variant of the usual potential (cf. [11, Theorem 4–11])

$$I_k\omega(x)=\sum_{i_1<\cdots<i_k}\sum_{\alpha=1}^{k}(-1)^{\alpha-1}\Big(\int_0^1 s^{k-1}f_{i_1\cdots i_k}(sx)ds\Big)x^{i_\alpha}$$
$$dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_\alpha}}\wedge\cdots\wedge dx^{i_k}.$$

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