23. *Certain Integral Operators**)

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1. Introduction. Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathcal{N}=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. For $f(z) \in \mathcal{A}(p)$, we define

$$I_0 f(z)=\left(\frac{f(z)}{z^p}\right)^{\alpha} \qquad (\alpha>0) \tag{1.2}$$

and

$$I_n f(z)=\frac{1}{z}\int_0^z I_{n-1}f(t)dt \qquad (n \in \mathcal{N}). \tag{1.3}$$

For $f(z)$ belonging to the class $\mathcal{A}(1)$, Thomas [4] has shown

Theorem A. *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\mathrm{Re}\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right\}>0 \qquad (z \in \mathcal{U}) \tag{1.4}$$

for some α $(\alpha>0)$, then

$$\mathrm{Re}\,(I_n f(z)) \geqq \gamma_n(r)>\gamma_n(1), \tag{1.5}$$

where $n \in \mathcal{N}_0=\{0,1,2,\cdots\}$ and

$$0<\gamma_n(r)=-1+2\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+\alpha)}<1. \tag{1.6}$$

Equality occurs for the function $f(z)$ defined by

$$f(z)=\left(\alpha\int_0^z t^{\alpha-1}\left(\frac{1-t}{1+t}\right)dt\right)^{1/\alpha}. \tag{1.7}$$

For $n=0$, (1.5) becomes

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\} \geqq \frac{\alpha}{r^{\alpha}}\int_0^r t^{\alpha-1}\left(\frac{1-t}{1+t}\right)dt \tag{1.8}$$

$$=-1+2\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k-1+\alpha},$$

which reduces to

$$-1+\frac{2}{r}\log(1+r)$$

when $\alpha=0$.

Also, Hallenbeck [1] has proved

Theorem B. *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\mathrm{Re}\,\{f'(z)\}>0 \qquad (z \in \mathcal{U}), \tag{1.9}$$

then

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(1.10) $$\operatorname{Re}\left(\frac{f(z)}{z}\right) \geqq -1+\frac{2}{r}\log(1+r) > -1+2\log 2.$$

Equality is attained for the function $f(z)$ *defined by*

(1.11) $$f(z) = -z+2\log(1+z).$$

Remark. Theorem A is a generalization of Theorem B.

Further, Owa and Obradović [3] have given

Theorem C. *If* $f(z) \in \mathcal{A}(1)$ *satisfies*

(1.12) $$\operatorname{Re}\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right\} > 0 \qquad (z \in \mathcal{U})$$

for some α $(\alpha>0)$, *then*

(1.13) $$\operatorname{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\} > \frac{1}{1+2\alpha} \qquad (z \in \mathcal{U}).$$

Remark 2. Theorem A is an improvement of Theorem C.

Some properties of I_n. We begin with the statement and the proof of the following result.

Theorem 1. *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

(2.1) $$\operatorname{Re}\left\{\frac{zf'(z)f(z)^{\alpha-1}}{z^{p\alpha}}\right\} > 0 \qquad (z \in \mathcal{U})$$

for some α $(\alpha>0)$, *then*

(2.2) $$\operatorname{Re}(I_n f(z)) \geqq \gamma_n(r) > \gamma_n(1),$$

where $n \in \mathcal{N}_0$ *and*

(2.3) $$0 < \gamma_n(r) = -1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)} < 1.$$

Equality in (2.2) *is attained for the function* $f(z)$ *given by*

(2.4) $$f(z) = \left(p\alpha\int_0^z t^{p\alpha-1}\left(\frac{1-t}{1+t}\right)dt\right)^{1/\alpha}.$$

Proof. Since the condition (2.1) implies that

(2.5) $$\operatorname{Re}\left\{\frac{f'(z)}{pz^{p-1}}\left(\frac{f(z)}{z^p}\right)^{\alpha-1}\right\} > 0 \qquad (z \in \mathcal{U}),$$

the function $h(z)$ defined by

(2.6) $$h(z) = \frac{f'(z)}{pz^{p-1}}\left(\frac{f(z)}{z^p}\right)^{\alpha-1}$$

satisfies $\operatorname{Re}(h(z))>0$ $(z \in \mathcal{U})$ and $h(0)=1$. It follows that

(2.7) $$\left(\frac{f(z)}{z^p}\right)^{\alpha} = \frac{p\alpha}{z^{p\alpha}}\int_0^z t^{p\alpha-1}h(t)dt,$$

that is, that

(2.8) $$\operatorname{Re}(I_0 f(z)) = \operatorname{Re}\left\{\left(\frac{f(z)}{z^p}\right)^{\alpha}\right\} = \operatorname{Re}\left\{\frac{p\alpha}{z^{p\alpha}}\int_0^z t^{p\alpha-1}h(t)dt\right\}.$$

Writing $z=re^{i\theta}$ and $t=\rho e^{i\theta}$ in (2.8), we have

$$\text{(2.9)}\qquad \operatorname{Re}(I_0f(z))=\frac{p\alpha}{r^{p\alpha}}\int_0^r \rho^{p\alpha-1}\operatorname{Re}\{h(\rho e^{i\theta})\}d\rho.$$

Note that the function $h(z)$ satisfying $\operatorname{Re}(h(z))>0$ $(z\in\mathcal{U})$ and $h(0)=1$ satisfies

$$\text{(2.10)}\qquad \operatorname{Re}(h(z))\geqq\frac{1-|z|}{1+|z|}\qquad (z\in\mathcal{U})$$

(cf. MacGregor [2, p. 532]). Thus (2.9) leads to

$$\begin{aligned}\text{(2.11)}\qquad \operatorname{Re}(I_0f(z))&\geqq\frac{p\alpha}{r^{p\alpha}}\int_0^r \rho^{p\alpha-1}\left(\frac{1-\rho}{1+\rho}\right)d\rho\\ &=-1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k-1+p\alpha}\\ &=\gamma_0(r).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\text{(2.12)}\qquad \operatorname{Re}(I_1f(z))&=\operatorname{Re}\left\{\frac{1}{z}\int_0^z I_0f(t)dt\right\}\\ &=\frac{1}{r}\int_0^r \operatorname{Re}\{I_0f(\rho e^{i\theta})\}d\rho\\ &\geqq\frac{1}{r}\int_0^r\left(-1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\rho^{k-1}}{k-1+p\alpha}\right)d\rho\\ &=-1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k(k-1+p\alpha)}\\ &=\gamma_1(r).\end{aligned}$$

Therefore, using the mathematical induction, we see that

$$\begin{aligned}\text{(2.13)}\qquad \operatorname{Re}(I_nf(z))&\geqq-1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)}\\ &=\gamma_n(r).\end{aligned}$$

Let the function $\phi_n(r)$ be defined by

$$\text{(2.14)}\qquad \phi_n(r)=p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)}\qquad (0<r<1).$$

Then $\phi_n(r)$ is absolutely convergent for n $(n\in\mathcal{N}_0)$ and for r $(0<r<1)$. Thus the suitably rearranging pairs of terms in $\phi_n(r)$ give that $1/2<\phi_n(r)<1$. This also gives that $0<\gamma_n(r)<1$. Further, since

$$\text{(2.15)}\qquad r\phi_n(r)=\int_0^r \phi_{n-1}(\rho)d\rho\qquad (n\in\mathcal{N}),$$

we have that $\phi_n'(r)<0$ and $\gamma_n(r)$ decreases with r as r tends to 1 for fixed n, and increases to 1 when $n\to\infty$ for fixed r. This completes the proof of Theorem 1.

Remark. If we take $p=1$ in Theorem 1, then we have Theorem A by Thomas [4].

Letting $\alpha=1/p$, Theorem 1 leads to

Corollary 1. *If $f(z)\in\mathcal{A}(p)$ satisfies*

$$\operatorname{Re}(f'(z)f(z)^{1/p-1})>0\qquad (z\in\mathcal{U}),$$

then

$$\operatorname{Re}(I_n f(z)) \geqq \gamma_n(r) > \gamma_n(1),$$

where $n \in \mathcal{N}_0$, $I_0 f(z) = f(z)^{1/p}/z$, *and*

$$0 < \gamma_n(r) = -1 + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}} < 1.$$

Equality is attained for the function $f(z)$ *defined by*

$$f(z) = (-z + 2\log(1+z))^p.$$

Taking $p=1$ in Corollary 1, we have

Corollary 2. *If* $f(z) \in \mathcal{A}(1)$ *satisfies* $\operatorname{Re}(f'(z)) > 0$ $(z \in \mathcal{U})$, *then*

$$\operatorname{Re}(I_n f(z)) \geqq \gamma_n(r) > \gamma_n(1),$$

where $n \in \mathcal{N}_0$, $I_0 f(z) = f(z)/z$, *and*

$$0 < \gamma_n(r) = -1 + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}} < 1.$$

Equality is attained for the function $f(z)$ *given by*

$$f(z) = -z + 2\log(1+z).$$

Remark. When $n=0$ in Corollary 2, we have Theorem B by Hallenbeck [1].

Further, making $\alpha=1$ in Theorem 1, we have

Corollary 3. *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0 \qquad (z \in \mathcal{U}),$$

then

$$\operatorname{Re}(I_n f(z)) \geqq \gamma_n(r) > \gamma_n(1),$$

where $n \in \mathcal{N}_0$, $I_0 f(z) = f(z)/z^p$, *and*

$$0 < \gamma_n(r) = -1 + 2p\sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n(k-1+p)} < 1.$$

Equality is attained for the function $f(z)$ *given by* (2.4) for $\alpha=1$.

3. Integral operator J_n. Next, for $f(z)$ in $\mathcal{A}(p)$, we introduce

$$J_0 f(z) = \frac{f(z)}{z^p} \tag{3.1}$$

and

$$J_n f(z) = \frac{a+1}{z^{a+1}} \int_0^z t^a J_{n-1} f(t)\,dt \qquad (n \in \mathcal{N}), \tag{3.2}$$

where $a > -1$.

For the above integral operator J_n, we derive

Theorem 2. *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \alpha \qquad (z \in \mathcal{U}), \tag{3.3}$$

where $\alpha < 1$, *then*

$$\operatorname{Re}(J_n f(z)) \geqq \gamma_n(r)\gamma_n(1), \tag{3.4}$$

where $n \in \mathcal{N}_0$ *and*

$$0 < \gamma_n(r) = 1 + 2(a+1)^n(1-\alpha)\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+a+1)^n} < 1. \tag{3.5}$$

Equality in (3.4) *is attained for the function* $f(z)$ *defined by*

(3.6) $$f(z)=\alpha z^p+(1-\alpha)z^p\left(\frac{1-z}{1+z}\right).$$

Proof. For $n=0$, (3.4) is trivial. For $n=1$, we have

(3.7) $$\begin{aligned}\frac{\partial}{\partial r}\left(\int_0^z t^a J_0 f(t)dt\right)&=\frac{\partial}{\partial r}\left(\int_0^z t^{a-p}f(t)dt\right)\\&=z^a\left(\frac{f(z)}{z^p}\right)e^{i\theta}\\&=z^a e^{i\theta}(\alpha+(1-\alpha)h(z)),\end{aligned}$$

where $z=re^{i\theta}$ and $h(z)=f(z)/z^p$. Since $\mathrm{Re}\,(h(z))\geqq(1-\rho)/(1+\rho)$ $(0\leqq\rho<1)$, for $a>-1$,

(3.8) $$\begin{aligned}\mathrm{Re}\,(J_1 f(z))&=\mathrm{Re}\left\{\frac{a+1}{z^{a+1}}\int_0^z t^a J_0(t)dt\right\}\\&\geqq\frac{a+1}{r^{a+1}}\int_0^r \rho^a\left(\alpha+(1-\alpha)\left(\frac{1-\rho}{1+\rho}\right)\right)d\rho\\&=\frac{a+1}{r^{a+1}}\int_0^r \rho^a\left(1+2(1-\alpha)\sum_{k=1}^{\infty}(-\rho)^k\right)d\rho\\&=1+2(a+1)(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{k+a+1}.\end{aligned}$$

Thus (3.4) holds true for $n=1$.

Further, assuming that (3.4) holds true for any n, and letting $t=\rho e^{i\theta}$, we have

(3.9) $$\begin{aligned}\mathrm{Re}\,(J_{n+1}f(z))&=\mathrm{Re}\left\{\frac{a+1}{z^{a+1}}\int_0^z t^a J_n f(t)dt\right\}\\&=\frac{a+1}{r^{a+1}}\int_0^r \rho^a\,\mathrm{Re}\,\{J_n f(\rho e^{i\theta})\}d\rho\\&\geqq\frac{a+1}{r^{a+1}}\int_0^r\left(\rho^a+2(a+1)^n(1-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^k\rho^{k+a}}{(k+a+1)^n}\right)d\rho\\&=\gamma_{n+1}(r).\end{aligned}$$

Also, we see that $0<\gamma_n(r)<1$ which completes the assertion of Theorem 2.

Taking $\alpha=p/(p+\beta)$, $\beta>0$, in Theorem 2, we have

Corollary 4. *If $f(z)\in\mathcal{A}(p)$ satisfies*

(3.10) $$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\}>\frac{p}{p+\beta}\qquad(z\in\mathcal{U}),$$

where $\beta>0$, then

(3.11) $$\mathrm{Re}\,(J_n f(z))\geqq\gamma_n(r)>\gamma_n(1),$$

where $n\in\mathcal{N}_0$ and

(3.12) $$0<\gamma_n(r)=1+2(a+1)^n\left(\frac{\beta}{p+\beta}\right)\sum_{k=1}^{\infty}\frac{(-r)^k}{(k+a+1)^n}<1.$$

Equality in (3.11) *is attained for the function $f(z)$ defined by*

(3.13) $$f(z)=\frac{1}{p+\beta}\left(pz^p+\beta z^p\left(\frac{1-z}{1+z}\right)\right).$$

References

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