21. Prime Graph Components of the Simple Groups of Lie Type over the Field of Even Characteristic

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1. Introduction. Let G be a finite group and $\Gamma(G)$ be the prime graph of G: the vertices $V(\Gamma(G))=\pi(G)$, the set of all prime divisors of $|G|$, two points p, $r \in V(\Gamma(G))$ are joined by an edge if and only if G contains an element of order pr. Let $\{\pi_1, \pi_2, \cdots, \pi_t\}$ be the connected components of $\Gamma(G)$ and if $|G|$ is even, denote the component containing 2 by π_1. Williams [4] classified the number of connected components for the simple groups of Lie type over the field of odd characteristic, alternating groups and 26 sporadic simple groups. However his argument breaks down for the simple groups of Lie type over the field of even characteristic. The purpose of this note is to classify the number of connected components for the simple groups of Lie type over the field of even characteristic. Namely we have the following:

Theorem. *For the simple groups of Lie type over the field of even characteristic, the connected components are as shown in Tables I, II and III where q is a power of 2 and p is an odd prime.*

The significance of the result is found in [3, 5]. We adopt the notation of [4] for the finite simple groups.

Two components

Table I

Type	Factors for primes in π_1	Factors for primes in π_2
$A_{p-1}(q), (p,q)\neq(3,2), (3,4)$	$q, q^i-1, 1\leq i\leq p-1$	$q^p-1/(q-1)(q-1,p)$
$A_p(q), q-1 \mid p+1$	$q, q^{p+1}-1, q^i-1, 1\leq i\leq p-1$	$q^p-1/q-1$
$C_k(q), k=2^n$	$q, q^k-1, q^{2i}-1, 1\leq i\leq k-1$	q^k+1
$C_p(q), (q-1,p)=1$	$q, q^p+1, q^{2i}-1, 1\leq i\leq p-1$	$q^p-1/(q-1)$
$D_p(q), (q-1,p)=1$	$q, q^{2i}-1, 1\leq i\leq p-1$	$q^p-1/(q-1)$
$D_{p+1}(2)$,	$2, 2^{2i}-1, 1\leq i\leq p-1, 2^p+1, 2^{p+1}-1$	2^p-1
${}^2A_3(2^2)$,	$2, 3$	5
${}^2A_{p-1}(q^2)$	$q, q^i-(-1)^i, 1\leq i\leq p-1$	$q^p+1/(q+1)(q+1,p)$
${}^2A_p(q^2), q+1 \mid p+1$	$q, q^{p+1}-1, q^i-(-1)^i, 1\leq i\leq p-1$	$q^p+1/(q+1)$
${}^2D_k(q), k=2^n, n\geq 2$	$q, q^{2i}-1, 1\leq i\leq k-1$	q^k+1
${}^2D_{K+1}(2), k=2^n, n\geq 2$	$2, 2^{2i}-1, 1\leq i\leq k-1, 2^k-1, 2^{k+1}+1$	2^k+1
$G_2(q), q\equiv 1(3)$	q, q^2-1, q^3-1	q^2-q+1
$G_2(q), q\equiv -1(3)$	q, q^2-1, q^3+1	q^2+q+1
${}^3D_4(q^3)$	q, q^6-1	q^4-q^2+1
${}^2F_4(2)'$	$2, 3, 5$	13
$E_6(q), q\equiv 1(3)$	$q, q^5-1, q^8-1, q^{12}-1$	$(q^6+q^3+1)/3$
$E_6(q), q\equiv -1(3)$	$q, q^5-1, q^8-1, q^{12}-1$	q^6+q^3+1
${}^2E_6(q^2), q\equiv -1(3)$	$q, q^5+1, q^8-1, q^{12}-1$	$(q^6-q^3+1)/3$
${}^2E_6(q^2), q\equiv 1(3)$	$q, q^5+1, q^8-1, q^{12}-1$	q^6-q^3+1

Three components

Table II

Type	Factors for primes in π_1	π_3	π_3
$A_1(q)$	q	$q-1$	$q+1$
$A_2(2)$	2	3	7
${}^2A_5(2^2)$	2, 3, 5	7	11
$E_7(2)$	2, 3, 5, 7, 11, 13, 17, 19, 31, 43	73	127
${}^2F_4(q)$	q, q^4-1, q^3+1	$q^2+\sqrt{2}q^{3/2}+q+\sqrt{2}q^{1/2}+1$	$q^2-\sqrt{2}q^{3/2}+q-\sqrt{2}q^{1/2}+1$
$F_4(q)$	q, q^4-1, q^6-1	q^4+1	q^4-q^2+1

Four or five components

Table III

Type	Factors for primes in π_1	π_2	π_3	π_4	π_5
$A_2(4)$	2	3	5	7	
${}^2B_2(q), q=2^{2k+1}$	q	$q-1$	$q+r+1, r^2=2q$	$q-r+1, r^2=2q$	
$E_8(q), q\equiv 1, 4(5)$	$q, q^j-1, j=8, 10, 12, 14, 18$	$q^8+q^7-q^5-q^4-q^3+q+1$	$q^8-q^7+q^5-q^4+q^3-q+1$	$q^8-q^6+q^4-q^2+1$	q^8-q^4+1
$E_8(q), q\equiv 2, 3(5)$	$q, q^j-1, j=8, 10, 12, 14, 18$, $q^8-q^6+q^4-q^2+1$	$q^8+q^7-q^5-q^4-q^3+q+1$	$q^8-q^7+q^5-q^4+q^3-q+1$	q^8-q^4+1	

2. Sketch of the proof. Let G be one of the simple groups of Lie type over the field of even characteristic. The structure of the centralizers of involutions of G are given in [1]. Now one can know the primes which are contained in π_1. Next consider the orders of maximal tori which one can find in [2]. For example let $G=G_2(q)$, q even. Then G contains maximal tori $T(A_2)$ and $T(G_2)$ with $|T(A_2)|=q^2+q+1$ and $|T(G_2)|=q^2-q+1$. For a central involution t one has $|C_G(t)|=q^6(q^2-1)$ and π_1 contains all prime divisors of q^2-1. If $q\equiv 1(3)$, then $|T(A_2)|\equiv 0(3)$ and $\pi(T(A_2))$ is contained in π_1. On the other hand if $q\equiv -1(3)$, then $|T(G_2)|\equiv 0(3)$ and $\pi(T(G_2))$ is contained in π_1. This classifies the components of the prime graph $\Gamma(G_2(q))$, q even. Similar but more complicated argument can be applied for the groups in our theorem. Details will appear elsewhere.

References

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