16. A Subadjunction Formula and Moishezon Fourfolds Homeomorphic to P_C^4

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§0. **Introduction.** The purpose of the present paper is to report some partial solutions to the following conjectures. Details [5] will appear elsewhere.

Conjecture MP_n. *Any Moishezon complex manifold homeomorphic to* P_C^n *is isomorphic to* P_C^n.

Conjecture DP_n. *Any complex analytic (global) deformation of* P_C^n *is isomorphic to* P_C^n.

Conjecture MP_n has been settled by Hirzebruch-Kodaira [1] and Yau [10] when the manifold under consideration is *projective or Kählerian.*

Recently Kollár [2] and the author [3] solved (MP_3) in the affirmative, each supplementing the other. Peternell [6] [7] also asserts (MP_3).

(0.1) **Theorem** [2] [3]. *Any Moishezon threefold homeomorphic to* P_C^3 *is isomorphic to* P_C^3.

(0.2) **Theorem.** *Let X be a Moishezon manifold of dimension n. Assume that there is a line bundle L on X such that* $c_1(X)=dc_1(L)$ $(d\geq n+1)$, $h^0(X, O_X(L))\geq n+1$, *and* $\kappa(L)=n$. *If a complete intersection of general* $(n-1)$*-members of the complete linear system* $|L|$ *is nonempty outside the base locus* Bs $|L|$, *then X is isomorphic to* P_C^n.

(0.3) **Theorem.** *Let X be a Moishezon manifold homeomorphic to* P_C^n, *and L a line bundle on X with* $L^n=1$. *Assume* $h^0(X, O_X(L))\geq n+1$. *If a complete intersection of general* $(n-1)$*-members of* $|L|$ *is nonempty outside* Bs $|L|$, *then X is isomorphic to* P_C^n.

(0.4) **Theorem.** *Let X be a Moishezon fourfold, and L a line bundle on X. Assume that* Pic $X=ZL$, $c_1(X)=dc_1(L)$ $(d\geq 5)$ *and* $h^0(X, O_X(L))\geq 5$. *Then X is isomorphic to* P_C^4.

(0.5) **Theorem.** *Let X be a Moishezon fourfold homeomorphic to* P_C^4, *and L a line bundle on X with* $L^4=1$. *Assume* $h^0(X, O_X(L))\geq 3$. *Then X is isomorphic to* P_C^4.

(0.6) **Corollary.** *Any complex analytic (global) deformation of* P_C^4 *is isomorphic to* P_C^4.

§1. **A complete intersection *l* and a subadjunction formula.** (1.1) Let X be a compact complex manifold of dimension n, a line bundle L on X with $h^0(X, O_X(L))\geq n-1$. Let V be a linear subspace of $H^0(X, L)$ of dimension $n-1$, $l:=l_V$ a scheme-theoretic complete intersection associated with V. More precisely, the ideal sheaf of O_X defining l is given by $I_l=\sum_{s\in V} sO_X$.

(1.2) **Lemma.** *Assume* $c_1(X)=dc_1(L)$. *Let* C *be an irreducible curve-component of* l_Y *along which* l_Y *is reduced generically. If* $d\geq n+1$, *and if* $LC\geq 1$, *then* $d=n+1$, $LC=1$, $C\simeq \boldsymbol{P}^1$, $N_{C/X}\simeq O_C(1)^{\oplus(n-1)}$ *and* C *is a connected component of* l_Y. *Moreover if* C *is not contained in* $\mathrm{Bs}|L|$, *then* $C\cap \mathrm{Bs}|L|$ *consists of at most one point.*

(1.3) **Theorem** (Subadjunction formula). *Let* X *be a compact complex manifold of dimension* n, D_i *a reduced irreducible divisor of* X $(1\leq i\leq m)$. *Assume that the scheme-theoretic complete intersection* $\tau=D_1\cap\cdots\cap D_m$ *has an irreducible component* $Z=Z_{\mathrm{red}}$ *of dimension* $n-m$ *along which* τ *is reduced generically. Let* $\nu: Y\to Z$ *be the normalization of* Z. *Then there exists an effective Weil divisor* Δ *of* Y *such that*

(1.3.1) $K_Y=\nu^*(K_X+D_1+\cdots+D_m)-\Delta$,

(1.3.2) *supp* $(\nu_*\Delta)$ *is the union of all the Weil divisors of* Z *whose supports are contained in either* $\mathrm{Sing}\,Z$ *or one of the irreducible components of* τ *other than* Z.

The condition (1.3.2) implies that *supp* $\Delta=\phi$ if and only if Z is smooth in codimension one and moreover Z intersect the irreducible components of τ other than Z along some subvarieties of at most $(n-m-2)$ dimension.

§2. **Proof of (0.5).** (2.1) **Lemma.** *Under the assumptions in* (0.5), *let* D *and* D' *be distinct members of* $|L|$, τ *the scheme-theoretic complete intersection* $D\cap D'$. *Then we have*

(2.1.1) $\mathrm{Pic}\,X=\boldsymbol{Z}L$, $K_X\simeq -5L$,

(2.1.2) $H^0(\tau, O_\tau)\simeq \boldsymbol{C}$,

(2.1.3) $|L|_\tau=|L_\tau|$.

(2.2) **Lemma.** *Let* D *and* D' *be general members of* $|L|$, *and* $\tau=D\cap D'$. *Let* $Z=Z_{\mathrm{red}}$ *be an irreducible component of* τ *along which* τ *is reduced generically. If* $Z\not\subset \mathrm{Bs}|L|$, *then* $\tau\simeq Z\simeq \boldsymbol{P}^2$ *and* $L_\tau\simeq O_{\boldsymbol{P}^2}(1)$.

Proof. Let $\nu: Y\to Z$ be the normalization of Z, $f: S\to Y$ the minimal resolution of Y and let $g=\nu\cdot f$. Then there exist by (1.3) an effective Weil divisor Δ on Y, effective Cartier divisors E and G on S with no common components such that the canonical sheaves K_Y and K_S are given by

$$K_Y=O_Y(-3\nu^*L-\Delta),\qquad K_S=O_S(-3g^*L-E-G)$$

with $f_*(E)=\Delta$, $f_*(G)=0$. Let $\Sigma:=f^{-1}(\Delta)\cup g^{-1}(\mathrm{Sing}\,Z)$.

Since $h^0(X,L)\geq 3$ and $Z\not\subset \mathrm{Bs}|L|$, g^*L is effective. By $P_m(S)=0$, $S\simeq \boldsymbol{P}^2$ or S is ruled. Assume that S is ruled. Let $\pi: S\to W$ be a morphism of S onto a curve W with general fiber $F\simeq \boldsymbol{P}^1$. Let $H\in g^*|L|$. We note that $E_{\mathrm{red}}+G_{\mathrm{red}}\subset H_{\mathrm{red}}$ for general D and D'. We also have,

$$-2=K_SF+F^2=K_SF=-(3H+E+G)F.$$

It follows that $HF=0$, $(E+G)F=2$. However this contradicts $E_{\mathrm{red}}+G_{\mathrm{red}}\subset H_{\mathrm{red}}$. Therefore $S\simeq Y\simeq \boldsymbol{P}^2$ and $G=0$. Since $H_{\mathrm{red}}\geq E_{\mathrm{red}}$ and $K_S=-3H-E$, we see that $O_S(H)\simeq O_{\boldsymbol{P}^2}(1)$, $E=0$. Since $E=0$, Z has by (1.3) at worst isolated singularities.

There exists $D''\in|L|$ such that $g^*(Z\cap D'')=H$ by the choice of H. Let $l=D\cap D'\cap D''$ be a scheme-theoretic complete intersection, and $C=g(H)_{\mathrm{red}}$.

Since $g^*D''=H\simeq \boldsymbol{P}^1$ and g is an isomorphism on $S\backslash\Sigma$, we have $H\backslash\Sigma\simeq C\backslash g(\Sigma)$, so that l is reduced generically along C. C is isomorphic to $Z\cap D''$ on $(Z\backslash g(\Sigma))\cap D''$. Namely $I_C=\sqrt{I_C}=I_l$ along $C\cap(Z\backslash g(\Sigma))$. We have

$$1=(H^2)_S=(g^*(L)H)_S=(Lg_*(H))_X=(LC)_X.$$

Therefore we can apply (1.2) to X, C and l to infer that C is a connected component of l and that $l\simeq C\simeq \boldsymbol{P}^1$ along C. If $\operatorname{Sing}\tau_{\mathrm{red}}$ is nonempty, then $\operatorname{Sing}\tau_{\mathrm{red}}\subset \operatorname{Bs}|L|$. Hence $Z\cap\operatorname{Sing}\tau_{\mathrm{red}}\subset Z\cap D''$ $(\simeq C)$. Consequently $Z\cap \operatorname{Sing}\tau_{\mathrm{red}}\subset C$. As C is a connected component of l, this shows that *Z is a connected component of τ*. In fact, if not, there is an irreducible component Z' $(\neq Z)$ of τ meeting Z. Then we choose a point $p\in Z\cap Z'$. We note that $Z\cap Z'$ is finite by $E=0$. Hence since $p\in Z\cap\operatorname{Sing}\tau_{\mathrm{red}}\subset C$, $Z'\cap D''$ contains an irreducible component (a curve or a surface) of l meeting C. This contradicts that C is a connected component of l.

However $h^0(\tau, O_\tau)=1$ by (2.1). Hence $Z\simeq\tau_{\mathrm{red}}$. As τ is Gorenstein and reduced generically along Z, τ is reduced everywhere and $\tau\simeq Z$. Since a prime Cartier divisor C of Z is smooth, so is Z along C. As $\operatorname{Sing} Z\subset Z\cap \operatorname{Sing}\tau_{\mathrm{red}}\subset C$, it follows that Z is smooth everywhere. Thus we see $\boldsymbol{P}^2\simeq S\simeq Y\simeq Z\simeq\tau$. Q.E.D.

(2.3) **Completion of the proof of (0.5).** Bertini's theorem guarantees existence of $\tau=D\cap D'$ with a component Z of τ as in (2.2). By (2.1.3) and (2.2), $\operatorname{Bs}|L|_\tau=\operatorname{Bs}|L_\tau|=\operatorname{Bs}|O_{\boldsymbol{P}^2}(1)|=\emptyset$. We have also $h^0(X, L)=h^0(\tau, L_\tau)+2=5$ and $(L^4)_X=(H^2)_S=1$. Consequently $X\simeq \boldsymbol{P}^4$ by an easy argument. Q.E.D.

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