10. *Domains of Square Roots of Regularly Accretive Operators*

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1. Introduction. The purpose of this paper is to give a sufficient condition for the domain of the square root of a regularly accretive operator and that of its adjoint operator to be the same.

Let X and V be two Hilbert spaces with $V\subset X$. Let the inclusion from V into X be continuous, and let V be dense in X. We denote by (f, g) (resp. $(u, v)_V$) the inner product in X (resp. V) and put $\|f\|=(f,f)^{1/2}$ and $\|u\|_V=(u,u)_V^{1/2}$.

Let $a[u, v]$ be a bounded sesquilinear form on $V\times V$;

(1.1) $\qquad |a[u,v]|\leqq M\|u\|_V\|v\|_V, \qquad M>0,$ for any $u, v\in V$.

We suppose that $a[u, v]$ is strongly coercive;

(1.2) $\qquad \operatorname{Re} a[u,u]\geqq\delta\|u\|_V^2, \qquad \delta>0,$ for any $u\in V$.

Let A be the closed operator associated with the variational triple $\{V, X, a\}$, that is, $u\in V$ belongs to $D(A)$ (the domain of A) if and only if there exists $f\in X$ such that $a[u,v]=(f,v)$ for any $v\in V$, and we define $Au=f$. We call A a *regularly accretive operator*.

We define the adjoint form $a^*[u,v]$ by $a^*[u,v]=\overline{a[v,u]}$ for any $u, v\in V$. It is known that the closed operator associated with the variational triple $\{V, X, a^*\}$ is the adjoint operator A^* of A.

As is well known, we can construct the fractional power A^θ $(0\leqq\theta\leqq 1)$ of the regularly accretive operator A. Kato [3] showed that $D(A^\theta)=D(A^{*\theta})\subset V$ if $0\leqq\theta<1/2$. But generally $D(A^{1/2})=D(A^{*1/2})$ does not hold, for Mcintosh [7] gave a counterexample. On the other hand, Kato and Lions obtained the following results independently.

Theorem A (Kato [4], Lions [6]). *Each of the following condition is sufficient for $D(A^{1/2})=D(A^{*1/2})=V$.*

(i) *Both $D(A^{1/2})$ and $D(A^{*1/2})$ are oversets (or subsets) of V.*

(ii) *$D(A^\theta)=D(A^{*\theta})$ for $\theta=1/2$ or 1.*

(iii) *There exists a Hilbert space W which satisfies* (1) $W\subset X$, (2) *V is a closed subspace of $[X, W]_{1/2}$,* (3) *$D(A)\subset W$ and $D(A^*)\subset W$, where $[X, W]_\theta$ $(0\leqq\theta\leqq 1)$ denotes the complex interpolation space of X and W.*

Remark 1. Theorem A-(iii) is due only to Lions.

Remark 2. We may replace Theorem A-(ii) with $D(A^\theta)=D(A^{*\theta})$ for some θ with $1/2\leqq\theta\leqq 1$, because we have $[X, D(A^\theta)]_{1/(2\theta)}=D(A^{1/2})$.

In the next section we give another sufficient condition for $D(A^{1/2})=D(A^{*1/2})=V$.

2. Main result. The sesquilinear form $a[u, v]$ can be written

$$a=a_R+ia_I, \qquad a_R=\frac{1}{2}(a+a^*), \qquad a_I=\frac{1}{2i}(a-a^*),$$

where a_R and a_I are symmetric forms.

Let Λ be the associated operator with $\{V, X, a_R\}$. Then it is known that Λ is a positive self-adjoint operator satisfying $D(\Lambda^{1/2})=V$ (with the equivalent norm) and $a_R[u,u]=\|\Lambda^{1/2}u\|^2$ for $u\in V$. We note that

$$|a_I[u,v]|\leqq\frac{M}{\delta}\|\Lambda^{1/2}u\|\,\|\Lambda^{1/2}v\|, \qquad u,v\in V,$$

holds from (1.1) and (1.2). In order to obtain a sufficient condition for $D(A^{1/2})=D(A^{*1/2})=V$ we need a stronger estimate for a_I as follows.

Theorem 1. *Let* $0<\theta\leqq 1$. *Suppose that*

(2.1) $\quad |a_I[u,v]|\leqq M_1\|\Lambda^{\theta/2}u\|\,\|\Lambda^{\theta/2}v\|, \qquad M_1>0,$ *for any* $u,v\in V$.

Then we have for any σ *with* $0<\sigma<1-\theta/2$,

(2.2) $\quad D(\Lambda^\sigma)\subset D(A^\sigma),$

(2.3) $\quad \|A^\sigma u-\Lambda^\sigma u\|\leqq C\|\Lambda^{\sigma-(1-\theta)/2}u\|, \qquad C>0,$ *for any* $u\in D(\Lambda^\sigma)$.

If we replace A *with* A^*, (2.2) *and* (2.3) *remain valid.*

Proof. Our proof is a slight modification of Kato [3] who proved Theorem 1 when $\theta=1$.

There exists a bounded symmetric operator in X such that

(2.4) $$(Bu,v)=a_I[\Lambda^{-1/2}u, \Lambda^{-1/2}v], \qquad u,v\in X.$$

Then we have

(2.5) $$A=\Lambda^{1/2}(1+iB)\Lambda^{1/2},$$

(2.6) $$(A+\lambda)^{-1}=(\Lambda+\lambda)^{-1}+\frac{\Lambda^{1/2}}{\Lambda+\lambda}BD_\lambda\frac{\Lambda^{1/2}}{\Lambda+\lambda}, \qquad \lambda>0,$$

where D_λ is a bounded operator in X with $\|D_\lambda\|\leqq 1+\|B\|$. The proof of (2.4)–(2.6) is found in Kato [3].

Let $0<\sigma<1-\theta/2$. Now we shall show that for $u\in D(\Lambda^\sigma)$,

(2.7) $$w=\lim_{R\to\infty}\int_0^R\lambda^\sigma\{(A+\lambda)^{-1}-(\Lambda+\lambda)^{-1}\}u\,d\lambda$$

exists and that

(2.8) $$\|w\|\leqq C\|\Lambda^{\sigma-(1-\theta)/2}u\|, \qquad C>0.$$

Here and in the sequel we denote by C positive constants independent of u, v, λ, t, a and b which may differ from each other. From (2.1), (2.4) and (2.6) we have for any $v\in X$,

(2.9) $$|(\{(A+\lambda)^{-1}-(\Lambda+\lambda)^{-1}\}u,v)|\leqq\left|a_I\left[\Lambda^{-1/2}D_\lambda\frac{\Lambda^{1/2}}{\Lambda+\lambda}u, \frac{1}{\Lambda+\lambda}v\right]\right| \leqq C\left\|\frac{\Lambda^{1/2}}{\Lambda+\lambda}u\right\|\left\|\frac{\Lambda^{\theta/2}}{\Lambda+\lambda}v\right\|.$$

Let $0<a<b<\infty$. It follows from (2.9) and Schwarz' inequality that

$$\left|\int_a^b\lambda^\sigma(\{(A+\lambda)^{-1}-(\Lambda+\lambda)^{-1}\}u,v)d\lambda\right|^2 \leqq C\left(\int_a^b\lambda^{2\sigma+\theta-1}\left\|\frac{\Lambda^{1/2}}{\Lambda+\lambda}u\right\|^2d\lambda\right)\left(\int_a^b\lambda^{1-\theta}\left\|\frac{\Lambda^{\theta/2}}{\Lambda+\lambda}v\right\|^2d\lambda\right).$$

Let $\{E_t\}$ be the spectral resolution of $\varLambda$. Then we have

$$\int_a^b \lambda^{1-\theta}\left\|\frac{\varLambda^{\theta/2}}{\varLambda+\lambda}v\right\|^2 d\lambda \leqq \int_0^\infty \lambda^{1-\theta}d\lambda\int_0^\infty \frac{t^\theta}{(t+\lambda)^2}d_t\|E_t v\|^2$$
$$\leqq \int_0^\infty \left(\int_0^\infty \frac{\lambda^{1-\theta}}{(1+\lambda)^2}d\lambda\right)d_t\|E_t v\|^2 \leqq C\|v\|^2.$$

Hence we obtain

$$\left\|\int_a^b \lambda^\sigma\{(A+\lambda)^{-1}-(\varLambda+\lambda)^{-1}\}u\,d\lambda\right\|^2 \leqq C\int_a^b \lambda^{2\sigma+\theta-1}d\lambda\int_0^\infty \frac{t}{(t+\lambda)^2}d_t\|E_t u\|^2$$
$$\leqq C\int_0^\infty t^{2\sigma+\theta-1}F(t\,;a,b)d_t\|E_t u\|^2,$$

where
$$F(t\,;a,b)=\int_{a/t}^{b/t}\frac{\lambda^{2\sigma+\theta-1}}{(\lambda+1)^2}d\lambda.$$

Noting that $\lim_{a\to\infty}F(t\,;a,b)=0$ and $F(t\,;a,b)\leqq F(1\,;0,\infty)<\infty$ for $-1<2\sigma+\theta-1<1$ and that $D(\varLambda^\sigma)\subset D(\varLambda^{\sigma-(1-\theta)/2})$, we conclude from the bounded convergence theorem that (2.7) exists and that (2.8) holds.

On the other hand, it follows from the definition of fractional powers or the spectral resolution of $\varLambda$ that

$$\varLambda^\sigma u=\frac{\sin\pi\sigma}{\pi}\lim_{R\to\infty}\int_0^R \lambda^\sigma\{\lambda^{-1}-(\varLambda+\lambda)^{-1}\}u\,d\lambda. \tag{2.10}$$

It follows from (2.7) and (2.10) that

$$w'=\frac{\sin\pi\sigma}{\pi}\lim_{R\to\infty}\int_0^R \lambda^\sigma\{\lambda^{-1}-(A+\lambda)^{-1}\}u\,d\lambda \tag{2.11}$$

exists. Therefore we have $u\in D(A^\sigma)$ and $w'=A^\sigma u$ (see Kato [1]). Hence we have proved $D(\varLambda^\sigma)\subset D(A^\sigma)$. (2.3) follows from (2.7), (2.8), (2.10) and (2.11).

Similarly we get the statement for A^*. Q.E.D.

Combining Theorem A-(i) and Theorem 1, we get the following

Theorem 2. *Let* (2.1) *hold for some* θ *with* $0\leqq\theta<1$. *Then we have* $D(A^{1/2})=D(A^{*1/2})=V$.

Remark 3. Shimakura [9] treated another type of perturbation. He considered a not necessarily regularly accretive operator $A=\varLambda+K$ in the Hilbert space X where $\varLambda$ is a strictly positive self-adjoint operator with the domain $D(\varLambda)$ dense in X, and K is a linear operator whose domain $D(K)$ contains $D(\varLambda)$. He obtained $D(A^\theta)=D(\varLambda^\theta)$ for any θ with $0\leqq\theta\leqq1$, assuming that the resolvent $(A+\lambda)^{-1}$ and $(\varLambda+\lambda)^{-1}$ satisfy some conditions. We note that $D(A)=D(\varLambda)$ in his case. On the other hand, in Theorem 2 we have $D(A)\neq D(\varLambda)$ generally, although we restrict ourselves to the case of regularly accretive operators. Hence our result is different from Shimakura's result.

It is interesting to investigate whether Theorem 1 can be improved or not, that is, whether $D(A^\sigma)=D(A^{*\sigma})=D(\varLambda^\sigma)$ is valid or not for any σ with $0<\sigma<1-\theta/2$ under condition (2.1). The following gives an affirmative example to this problem. Let $I=(0,1)\subset\boldsymbol{R}$. Let $X=L_2(I)$ and $V=H^2(I)$

where $H^2(I)$ is the Sobolev space. For $\alpha \in C\backslash R$ let us put

$$a[u, v]=\int_I (u''(x)\overline{v''(x)}+\alpha u'(x)\overline{v'(x)})dx.$$

The domains of fractional powers of A, A^* and Λ are given in terms of the boundary conditions such as

$$u''(0)=u''(1)=0, \tag{2.12}$$

$$\int_0^1 \frac{|u''(x)|^2}{d(x)}dx<\infty, \tag{2.13}$$

$$u^{(3)}(0)-\alpha u'(0)=u^{(3)}(1)-\alpha u'(1)=0, \tag{2.14}$$

$$\int_0^1 \frac{|u^{(3)}(x)-\alpha u'(x)|^2}{d(x)}dx<\infty, \tag{2.15}$$

where $d(x)=\min\{|x|, |x-1|\}$. We put

$$E_\alpha=\{u \in H^4(I)\,;\ u \text{ satisfies (2.12) and (2.14)}\},$$

and obtain

$$D(A)=E_\alpha, \qquad Au=u^{(4)}-\alpha u^{(2)}. \tag{2.16}$$

For A^* (resp. Λ) we have (2.16) with α replaced by $\bar{\alpha}$ (resp. $\mathrm{Re}\,\alpha$). Clearly we have $D(A)\neq D(A^*)\neq D(\Lambda)\neq D(A)$. From the interpolation theorem and Grisvard [2, Theorem 8.1] it follows that

$$D(A^\sigma)=[L_2(I), D(A)]_\sigma$$

$$=\begin{cases} H^{4\sigma}(I) & (0<\sigma<\frac{5}{8}) \\ \{u \in H^{5/2}(I)\,;\ u \text{ satisfies (2.13)}\} & (\sigma=\frac{5}{8}) \\ \{u \in H^{4\sigma}(I)\,;\ u \text{ satisfies (2.12)}\} & (\frac{5}{8}<\sigma<\frac{7}{8}) \\ \{u \in H^{7/2}(I)\,;\ u \text{ satisfies (2.12) and (2.15)}\} & (\sigma=\frac{7}{8}) \\ \{u \in H^{4\sigma}(I)\,;\ u \text{ satisfies (2.12) and (2.14)}\} & (\frac{7}{8}<\sigma<1). \end{cases}$$

The domains of fractional powers of A^* and Λ are given in the similar way. Therefore it follows that

$$D(A^\sigma)=D(A^{*\sigma})=D(\Lambda^\sigma), \qquad \text{for } 0<\sigma<\frac{7}{8},$$

and

$$D(A^\sigma)\neq D(A^{*\sigma})\neq D(\Lambda^\sigma)\neq D(A^\sigma), \qquad \text{for } \frac{7}{8}\leqq\sigma\leqq 1.$$

On the other hand, we have for some $M_1>0$,

$$|a_I[u, v]|\leqq|\mathrm{Im}\,\alpha|\|u'\|_{L_2(I)}\|v'\|_{L_2(I)}\leqq M_1\|\Lambda^{1/4}u\|\|\Lambda^{1/4}v\|$$

where the last inequality is due to Lemma 3 in the next section. Thus this example suggests the possibility of an improvement of Theorem 1.

3. **Application.** We can apply Theorem 2 to the non-self-adjoint elliptic operator with non-smooth coefficients and a non-smooth boundary. Let m and n be positive integers. Let Ω be a bounded domain in R^n with the restricted cone property. Let $X=L_2(\Omega)$. Let V be the closed subspace of the Sobolev space $H^m(\Omega)$ including $H_0^m(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$). We denote by $\|\ \|_m$ the norm of $H^m(\Omega)$. Let $a[u, v]$ be an integro-differential sesquilinear form of order m with bounded coefficients;

$$a[u, v]=\int_\Omega \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x)D^\alpha u(x)\overline{D^\beta v(x)}dx, \qquad u, v \in V,$$

$$\alpha=(\alpha_1, \cdots, \alpha_n), \qquad D^\alpha=(-\sqrt{-1})^{|\alpha|}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n},$$

which satisfies (1.2). Let A and Λ be the operators as defined in the previous sections.

Lemma 3. *In the above situation we have*

$$\|D^\alpha u\|\leqq C\|\Lambda^{|\alpha|/2m}u\|, \qquad 0\leqq|\alpha|\leqq m,\ u\in V.$$

Proof. It follows from the complex interpolation theory that

$$H^k(\Omega)=[L_2(\Omega), H^m(\Omega)]_{k/m}\supset[L_2(\Omega), V]_{k/m}$$
$$=[L_2(\Omega), D(\Lambda^{1/2})]_{k/m}=D(\Lambda^{k/2m}), \qquad 0\leqq k\leqq m,$$

which gives the lemma. Q.E.D.

Theorem 4. *Suppose that*

$$\text{(3.1)} \qquad a_{\alpha\beta}=\overline{a_{\beta\alpha}} \qquad (|\alpha|+|\beta|=2m,\ 2m-1).$$

Then we have $D(A^{1/2})=D(A^{*1/2})=V$.

Proof. It follows from the assumption that

$$|a_I[u, v]|\leqq M_1\|u\|_{m-1}\|v\|_{m-1}, \quad M_1>0, \qquad \text{for any } u, v\in V.$$

Combining the above inequality and Lemma 3, we get (2.1) for $\theta=1-1/m$. Therefore we can apply Theorem 2 to obtain the theorem. Q.E.D.

We stress that the smoothness of the coefficients $a_{\alpha\beta}$ and the boundary $\partial\Omega$ are not assumed in Theorem 4. When the coefficients and the boundary are sufficiently smooth and when V satisfies some condition such as $V=H^m(\Omega)$ or $V=H_0^m(\Omega)$ etc., Lions [6] also obtained Theorem 4 without assuming (3.1) by using the relations $D(A)\subset H^{2m}(\Omega)$, $D(A^*)\subset H^{2m}(\Omega)$ and $[L_2(\Omega), H^{2m}(\Omega)]_{1/2}=H^m(\Omega)$, and applying Theorem A-(iii) with $W=H^{2m}(\Omega)$. We note that $D(A)\subset H^{2m}(\Omega)$ and $D(A^*)\subset H^{2m}(\Omega)$ do not always hold when the coefficients and the boundary are not smooth. It seems reasonable to conjecture that Theorem 4 is valid without assuming (3.1). However this question remains open.

Our result remains valid if $a[u, v]$ has some boundary integrals containing derivatives of order $\leqq m-1$ when $\partial\Omega$ is sufficiently smooth.

References

[1] D. Fujiwara: Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order. Proc. Japan Acad., **43**, 82–86 (1967).

[2] P. Grisvard: Caractérisation de quelques espaces d'interpolation. Arch. Rat. Mech. Anal., **25**, 40–63 (1967).

[3] T. Kato: Fractional powers of dissipative operators. J. Math. Soc. Japan, **13**, 246–274 (1961).

[4] ——: Fractional powers of dissipative operators. II. ibid., **14**, 242–248 (1962).

[5] ——: Perturbation Theory for Linear Operators. Grundlehren der mathematischen Wissenschaften, vol. 132, Springer (1980).

[6] J. L. Lions: Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs. J. Math. Soc. Japan, **14**, 233–241 (1962).

[7] A. Mcintosh: On the comparability of $A^{1/2}$ and $A^{*1/2}$. Proc. Amer. Math. Soc., **32**, 430–434 (1972).

[8] R. Seeley: Interpolation in L^p with boundary condition. Studia Math., **44**, 47–60 (1972).

[9] N. Shimakura: Sur les domaines de puissances fractionnaires d'opérateurs. Bull. Soc. Math. France, **96**, 265–288 (1968).