63. A Remark on Certain Analytic Functions

By Sang Keun LEE*) and Shigeyoshi OWA**)

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Abstract: A class $A_p(\alpha, \beta; a, b)$ of certain analytic functions in the unit disk, which is a generalization of the class of p-valently starlike functions of order α and of p-valently convex functions of order β, is introduced. The object of the present paper is to derive a property of the class $A_p(\alpha, \beta; a, b)$.

1. Introduction. Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N=\{1, 2, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$. A function $f(z) \in A_p$ is said to be p-valently starlike of order α if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z \in U) \tag{1.2}$$

for some α $(0\leq\alpha<p)$. We denote by $S_p^*(\alpha)$ the subclass of A_p consisting of functions which are p-valently starlike of order α in U. A function $f(z) \in A_p$ is said to be p-valently convex of order β if it satisfies

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta \qquad (z \in U) \tag{1.3}$$

for some β $(0\leq\beta<p)$. Also we denote by $K_p(\beta)$ the subclass of A_p consisting of all such functions.

Some subclasses of p-valent functions were recently studied by Nunokawa ([1], [2]), Owa ([3], [4]), Owa and Ren [5], Owa and Yamakawa [6], and Saitoh [7].

With the help of the classes $S_p^*(\alpha)$ and $K_p(\beta)$, we introduce the subclass $A_p(\alpha, \beta; a, b)$ of A_p consisting of functions which satisfy

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}-\alpha\right)^a\left(1+\frac{zf''(z)}{f'(z)}-\beta\right)^b\right\}>0 \qquad (z \in U) \tag{1.4}$$

for some α $(0\leq\alpha<p)$, β $(0\leq\beta<p)$, $a \in R$ and $b \in R$, where R means the set of all real numbers.

Note that $A_p(\alpha, \beta; 1, 0)=S_p^*(\alpha)$ and $A_p(\alpha, \beta; 0, 1)=K_p(\beta)$. Therefore $A_p(\alpha, \beta; a, b)$ is a generalization of $S_p^*(\alpha)$ and $K_p(\beta)$.

2. Main result. We begin with the statement and the proof the following result.

Main theorem. For $0\leq t\leq 1$, we have

$$A_p(\alpha, \beta; a, b)\cap S_p^*(\alpha)\subset A_p(\alpha, \beta; (a-1)t+1, bt).$$

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*) Department of Mathematics, Gyeongsang National University, Korea.

**) Department of Mathematics, Kinki University, Japan.

Proof. Define the function $V(z)$ by

$$V(z)=\left(\frac{zf'(z)}{f(z)}-\alpha\right)^a\left(1+\frac{zf''(z)}{f'(z)}-\beta\right)^b \tag{2.1}$$

for $f(z)\in A_p(\alpha, \beta; a, b)\cap S_p^*(\alpha)$. Then we see that $\mathrm{Re}\,(V(z))>0$ for all $z\in U$. Let

$$U(z)=\frac{zf'(z)}{f(z)}-\alpha. \tag{2.2}$$

Then $f(z)\in S_p^*(\alpha)$ implies that $\mathrm{Re}\,(U(z))>0$ for all $z\in U$. It follows from (2.1) and (2.2) that

$$\left(\frac{zf'(z)}{f(z)}-\alpha\right)^{(a-1)t+1}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right)^{bt}=(U(z))^{1-t}(V(z))^t. \tag{2.3}$$

Defining the function $F(z)$ by

$$F(z)=(U(z))^{1-t}(V(z))^t \qquad (0\leq t\leq 1), \tag{2.4}$$

we obtain that

$$F(0)=(p-\alpha)^{(a-1)t+1}(p-\beta)^{bt}>0 \tag{2.5}$$

and

$$\begin{aligned} |\arg\,(F(z))| &= |\arg\,((U(z))^{1-t}(V(z))^t)| \\ &\leq (1-t)|\arg\,(U(z))|+t|\arg\,(V(z))| \\ &\leq \pi/2. \end{aligned} \tag{2.6}$$

This shows that $\mathrm{Re}\,(F(z))>0$ $(z\in U)$, that is, $f(z)\in A_p(\alpha, \beta; (a-1)t+1, bt)$.

By virtue of our main theorem, we state

Conjecture. For $0\leq t\leq 1$, we have

$$A_p(\alpha, \beta; a, b)\subset A_p(\alpha, \beta; (a-1)t+1, bt).$$

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