56. *On Smooth Quartic Embedding of Kummer Surfaces*

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1991)

1. The purpose of the note is to show the following fact: *For any abelian surface admitting a polarization with the reduced pfaffian three, one can always construct a birational morphism of the associated Kummer surface into $P_3(C)$ whose image is a quartic surface. The morphism is a smooth embedding if the abelian surface can not be principally polarized.*

We will also discuss some geometry around this fact. Let A be an abelian surface, E the universal cover of A and G the lattice such $A=E/G$. Suppose that a polarization (ample line bundle) H is given to A. We identify H with its Riemann form. (See Weil [4].) H is thus a hermitian form on E whose imaginary part is $\boldsymbol{Z}$-valued over G. We assume that the reduced pfaffian of H is three, that is, that the determinant of the imaginary part over G is equal to nine. We denote the space of *odd* theta functions of type $(2H, \mathbf{1})$ ($\mathbf{1}$: the trivial semi-character) by V. V is then four-dimensional. Since the semi-character is trivial, the theta functions in V necessarily vanish at the two-torsion points of A. Regarding V as a subspace of $\Gamma(A, H)$ we obtain a rational mapping of A into $\boldsymbol{P}(V^*)\simeq \boldsymbol{P}_3(\boldsymbol{C})$. ($V^*$ is the dual space of V. For a complex vector space W we denote by $\boldsymbol{P}(W)$ the projective space $(W\backslash\{0\})/\boldsymbol{C}^\times$.) Let S be the Kummer surface associated with A i.e. the minimal desingularization of the quotient of A by the involution $z\leftrightarrow -z$. Since all elements in V are *odd*, the rational mapping induces a rational mapping of S into $\boldsymbol{P}(V^*)$ and we see that this mapping is actually a birational morphism and that the image is a quartic surface. The line bundle $2H$ induces a line bundle over S, which we denote by $(2H)$. This is of self intersection twelve and is orthogonal to the exceptional divisors E_i $(i=1, 2, \cdots, 16)$ of the desingularization S. The line bundle

$$L:=(2H)-\frac{1}{2}\sum_{i=1}^{16} E_i$$

gives the quartic polarization of S. Since $(L, E_i)=1$, the image of E_i is a line, which we denote by l_i. We have thus obtained the sixteen lines on the image of S which are disjoint if $S\to \boldsymbol{P}(V^*)$ is an embedding. From now on we assume that A does not have a principal polarization; we identify S with its image in $\boldsymbol{P}(V^*)$.

2. The geometry of quartic surface S is quite interesting. Besides l_i $(i=1, 2, \cdots, 16)$ there is another class of disjoint sixteen lines. They are obtained in the following way: *Let* ψ_i $(i=1, 2, \cdots, 16)$ be the semi-characters on G with respect to H whose values lie in $\{\pm 1\}$. By W_i we denote the space of theta functions of type (H, ψ_i). W_i is of dimension three. Since ψ_i is $\{\pm 1\}$-valued, the involution $z \leftrightarrow -z$ of E induces an involution of W_i, which we denote by ι_i. W_i splits into the sum of two eigen spaces W_i', W_i'' of ι_i which are uniquely determined by the convention $\dim W_i'=1$, $\dim W_i''=2$. We fix a basis $\{\theta_i^{(0)}, \theta_i^{(1)}, \theta_i^{(2)}\}$ of W_i in such a way that $\theta_i^{(0)} \in W_i'$; $\theta_i^{(1)}, \theta_i^{(2)} \in W_i''$. Let $\tilde{r}_i$ be the zero locus of $\theta_i^{(0)}$ in A. By Riemann-Roch $\tilde{r}_i$ is a smooth curve of genus four. We see that there are exactly ten two-torsion points of A on $\tilde{r}_i$; so its image on S, denoted by r_i, is a rational curve; it is actually a line since we can check that $(L, r_i)=1$. The lines r_i $(i=1, 2, \cdots, 16)$ are disjoint. We see also that, for each l_j, there are exactly ten r_i intersecting l_j. When we regard $\theta_i^{(0)}\theta_i^{(1)}, \theta_i^{(0)}\theta_i^{(2)} \in V$ as linear forms on V^*, the line r_i is defined by $\theta_i^{(0)}\theta_i^{(1)}=\theta_i^{(0)}\theta_i^{(2)}=0$ in $\boldsymbol{P}(V^*)$.

3. The rational function $\theta_i^{(1)}/\theta_i^{(2)}$ induces a pencil of genus 4 curves on A; the base points of the pencil are exactly the two-torsions of A not lying on $\tilde{r}_i$. Thus this induces a pencil of elliptic curves on S. Since S is smooth, S is an elliptic surface in the sense of Kodaira [1] with respect to the morphism $\pi_i: X \to \boldsymbol{P}(W_i' \otimes W_i'') \simeq \boldsymbol{P}_1(\boldsymbol{C})$; π_i is induced by $W_i' \times W_i'' \ni (\theta', \theta'') \mapsto \theta'\theta'' \in V$. We want to characterize the singular fibres of π_i. Recall that the pencil of planes passing through r_i induces this fibration: Since S is quartic, the section of S by the generic plane of the pencil decomposes to r_i and a smooth cubic curve which is a regular fibre of π_i. Now suppose that some l_j intersects r_i and consider the plane passing through both lines. Then, for this special member of the pencil, the above plane cubic decomposes into l_j and the complementary conic. This gives us a singular fibre of type I_2 (under the genericity assumption mentioned later). We obtain ten singular fibres of type I_2.

4. To characterize the remaining singular fibres of π_i, we begin with the following remark. There are exactly four isogenies $h_k: A_k \to A$ $(k=1, 2, 3, 4)$ such that $h_k^*(H)$ is three times the principal polarization H_k of A_k. Since the degree of h_k is three, ψ_i $(i=1, 2, \cdots, 16)$ can also be regarded as the $\{\pm\}$-semi-characters of (A_k, H_k). The theta function $\vartheta_{k,i}$ of type (H_k, ψ_i) is unique up to a multiplicative constant and $\vartheta_{k,i}^3 \in W_i$. We see easily that $\vartheta_{k,i}^3 \in W_i''$, we see further that the plane section $\theta_i^{(0)}\vartheta_{k,i}^3=0$ is a singular fibre of type I_1 under the genericity assumption:

(GA). *There is no six-torsion point $p \in A_k$ on the theta divisor $\vartheta_{k,i}=0$ such that $h_k(2p)=0$.*

In fact under this condition, the image by h_k of theta divisor $\vartheta_{k,i}=0$ is a genus two curve with two nodes which are transposed by the involution $z \leftrightarrow -z$. Its further image on S is a rational curve with a node.

Remark. If there is such a six-torsion p as in (GA) on $\vartheta_{k,i}=0$, then $h_k(p)$ is a tac-node of the image of the theta divisor, it is also a two-torsion of A. Thus, in this case, the two fibres of types I_2 and I_1 glue together into one fibre of type III. This fibre consists of some l_j intersecting r_i and a conic tangent to it. We have seen that III is the only type of degeneration of singular fibres so far as $S \to \boldsymbol{P}(V^*)$ is a smooth embedding. It is an interesting problem to list up all singular images of $S \to \boldsymbol{P}(V^*)$.

Example. To close this note we will give a beautiful smooth quartic surface with the maximal degeneration. It is the zero locus in $\boldsymbol{P}_3(\boldsymbol{C})$: (x_1, x_2, x_3, x_4) of

$$5(x_1^4+x_2^4+x_3^4+x_4^4)+6(x_1^2x_2^2+x_1^2x_3^2+x_1^2x_4^2+x_2^2x_3^2+x_2^2x_4^2+x_3^2x_4^2)+8\sqrt{-15}\,x_1x_2x_3x_4.$$

This surface has the symmetry of the Weyl group of D_4 (generated by permutations and even sign changes of coordinates). Clearly the line r_1: $\sqrt{-15}x_1+x_2+x_3+x_4=x_2+\omega^2x_3+\omega x_4=0$ $(\omega^2+\omega+1=0)$ is on the surface; $r_2, \cdots, r_{16}$ are obtained by the transformation of even elements in the Weyl group and $l_1, l_2, \cdots, l_{16}$ are obtained by odd elements of the group. In the elliptic fibration associated with r_1 there are four singular fibres of type III and six singular fibres of type I_2. In this case, the abelian surface A is covered by the product of elliptic curves corresponding to the two ideal classes of $\boldsymbol{Q}(\sqrt{-15})$; the covering degree is twelve.

Mizukami [2] showed in the case where A is a product of elliptic curves which are two-isogenous, that the corresponding Kummer surface admits a smooth quartic embedding. The attempt of looking for another class of such Kummer surfaces was naturally a part of motivation to the present work.

References

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