53. *On Solutions of the Poincaré Equation*

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1. Introduction and result. Consider a map $F: C^2 \to C^2$ defined by

(1) $$F: {}^t(x, y) \longmapsto {}^t(y, ax+p(y)),$$

where a is a nonzero constant and $p(y)$ is a polynomial of degree $d \geq 2$. The map F is called a *twisted elementary map* (Kimura [2]). We denote by F^k the k-times iteration of F. Assume that $z_0 = {}^t(x_0, y_0) \in C^2$ is a periodic point of F of period k, i.e. a fixed point of F^k. Let J be the Jacobian matrix of F^k at z_0. Let ρ be an eigenvalue of J, $v = {}^t(v_1, v_2) \in C^2$ an eigenvector of J corresponding to the eigenvalue ρ. The eigenvalue ρ is said to be *unstable* (resp. *stable*) if $|\rho| > 1$ (resp. if $|\rho| < 1$).

Definition (Kimura [2]). Suppose that ρ is unstable (resp. stable). A holomorphic map $\Xi: C \to C^2$ is called an unstable (resp. a stable) curve through z_0 if the following two conditions hold:

(2) $$\Xi(\rho t) = F^k(\Xi(t)) \qquad \text{for } t \in C$$

(3) $$\Xi(t) = z_0 + vt + O(t^2) \qquad \text{as } t \longrightarrow 0.$$

If none of ρ^n $(n = 2, 3, 4, \cdots)$ is an eigenvalue of J, it is known that there exists an unstable (a stable) curve through z_0 ([2]). The functional equation (2) is called the *Poincaré equation*, since Poincaré [3] was the first to consider this type of functional equation (cf. Dixon-Esterle [1]). In this paper we shall establish the following:

Main theorem. *Each component of the (un) stable curve $\Xi(t)$ is an entire function of order τ and of finite type, where τ is given by*

$$\tau = \frac{\log d}{|\log |\rho|^{1/k}|}.$$

Remark. In a special case $k=1$, the result is already shown in [2]. As we shall see below, however, we require much subtler estimates than those in [2] to establish the theorem for $k > 1$.

2. Notation. Throughout this paper we employ the following notation.

(a) Let $\Xi_m = {}^t(\xi_m, \eta_m): C \to C^2$ be holomorphic maps defined recursively by $\Xi_0(t) = \Xi(t)$ and

(4) $$\Xi_m(t) = F(\Xi_{m-1}(\lambda^{-1} t)) \qquad \text{for } m \in Z,$$

where $\lambda = \rho^{1/k}$. We put $\xi = {}^t(\xi_0, \cdots, \xi_{k-1})$ and $\eta = {}^t(\eta_0, \cdots, \eta_{k-1})$.

(b) For a k-vector $u = {}^t(u_0, \cdots, u_{k-1}) \in C^k$, we put $\|u\| = |u_0| + \cdots + |u_{k-1}|$ and $p(u) = {}^t(p(u_0), \cdots, p(u_{k-1}))$.

(c) We put for $r > 0$,

$$M_m(r)=\max_{|t|=r}|\xi_m(t)|+1, \qquad N_m(r)=\max_{|t|=r}|\eta_m(t)|+1,$$
$$M(r)=\max_{|t|=r}\|\xi(t)\|, \qquad N(r)=\max_{|t|=r}\|\eta(t)\|,$$

(d) We denote by C_j various *positive* constants depending only on $a, p(y)$ and k.

3. Lemmata. We shall give a proof of Main Theorem only in the unstable case $|\rho|>1$; we can treat the stable case in a similar manner. In order to establish Main Theorem, we shall show the following lemmata successively.

Lemma 1. $\log M_0(r),\ \log N_0(r)\leq C_0r^\tau+C_1$.

Lemma 2. $\sum_{j=0}^{k-1}\log M_j(r)\geq C_2r^\tau-C_3$.

Lemma 3. $\log M_0(r)+\log N_0(r)\geq C_4r^\tau-C_5$.

Lemma 4. $\log M_0(r),\ \log N_0(r)\geq C_6r^\tau-C_7$.

Main Theorem is an easy consequence of Lemma 1 and Lemma 4.

4. Proof of Lemma 1. Put $F^k(x,y)={}^t(f(x,y),\ g(x,y))$. It is easy to see that $f(x,y)$ and $g(x,y)$ are polynomials of degree d^{k-1} and d^k, respectively. Hence we have $1+|f(x,y)|,\ 1+|g(x,y)|\leq C_0(2+|x|+|y|)^{d^k}$. Since $\Xi(t)$ is an unstable curve, we have $\xi_0(\rho t)=f(\xi_0(t),\eta_0(t))$ and $\eta_0(\rho t)=g(\xi_0(t),\eta_0(t))$. Substituting these into the above inequality, we obtain $M_0(|\rho|r),\ N_0(|\rho|r)\leq C_0\{M_0(r)+N_0(r)\}^{d^k}$. So, letting $S(r)=M_0(r)+N_0(r)$, we have

$$S(|\rho|r)\leq \exp\{(d^k-1)C_1\}S(r)^{d^k}.$$

We see that $s(r)=\exp(C_2r^\tau-C_1)$ satisfies $s(|\rho|r)=\exp\{(d^k-1)C_1\}s(r)^{d^k}$. Assume that C_2 is so large that $S(r)\leq s(r)$ for $1\leq r\leq|\rho|$. Then it is easy to see that $S(r)\leq s(r)$ for $r\geq 1$. Hence we have $S(r)\leq\exp(C_2r^\tau+C_3)$ for $r\geq 0$. This shows that $\log M_0(r),\ \log N_0(r)\leq C_2r^\tau+C_3$, which establishes Lemma 1.

5. Proof of Lemma 2. By (4), $\Xi_m(t)$ is an unstable curve through $z_m=F^m(z_0)$. We see that $\Xi_k(t)=\Xi_0(t)$. Hence it follows from (4) that $\xi(\lambda t)=A\eta(t)$ and $\eta(\lambda t)=aA\xi(t)+p(A\eta(t))$, where $A=(a_{ij})$ is a $k\times k$ permutation matrix defined by $a_{ij}=1$ if $i-j\equiv 1\ (\mathrm{mod}\ k)$, $a_{ij}=0$ otherwise. Eliminating η, we obtain

$$A^{-1}\xi(\lambda t)=p(\xi(t))+aA\xi(\lambda^{-1}t). \tag{5}$$

Since $p(y)$ is a polynomial of degree d, we have $|p(y)|\geq C_0|y|^d-C_1$. Applying this estimate to (5), we obtain

$$\|\xi(\lambda t)\|\geq C_0\|\xi(t)\|^d-\|\xi(\lambda^{-1}t)\|-C_2. \tag{6}$$

Since $\xi_j(t)$ $(j=0,1,\cdots,k-1)$ are entire functions not identically zero, $\|\xi(t)\|$ is a subharmonic function. Hence $M(r)$ is monotonically increasing in r and tends to $+\infty$ as $r\to\infty$. Thus (6) implies that $M(|\lambda|r)\geq C_0M(r)^d-C_1M(r)-C_2$. If r is sufficiently large, then so is $M(r)$. Thus we may assume that

$$M(|\lambda|r)\geq\exp\{(d-1)C_3\}M(r)^d,\quad M(r)\geq 2\quad\text{for } r\geq r_0.$$

We see that $m(r)=\exp(C_4r^\tau-C_3)$ satisfies $m(|\lambda|r)=\exp\{(d-1)C_3\}m(r)^d$. Assume that C_4 is so small that $M(r)\geq m(r)$ for $r_0\leq r\leq|\lambda|r_0$. We can easily show that $M(r) \geq m(r)$ for $r \geq r_0$. Hence we have

$$\log M(r) \geq C_4 r^\tau - C_1. \tag{7}$$

By the definition of $M(r)$ and $M_j(r)$, it is evident that $M(r) \leq \sum_{k=0}^{k-1} M_j(r)$. On the other hand, the following inequality holds for $x_j \geq 1$,

$$\sum_{j=0}^{k-1} \log x_j \geq \log\left(\sum_{j=0}^{k-1} x_j\right) - k.$$

Combining these inequalities with (7), we obtain $\sum_{j=0}^{k-1} \log M_j(r) \geq C_4 r^\tau - C_3$. Lemma 2 is thus established.

6. Proof of Lemma 3. We rewrite (4) as

$$\xi_m(\lambda t) = \eta_{m-1}(t), \qquad \eta_m(\lambda t) = p(\eta_{m-1}(t)) + a\xi_{m-1}(t). \tag{8}$$

Eliminating η, we obtain $\xi_{m+1}(\lambda t) = p(\xi_m(t)) + a\xi_{m-1}(\lambda^{-1}t)$. If we put $\theta_m(t) = \xi_m(\lambda^m t)$, then we have $\theta_{m+1} = p(\theta_m) + a\theta_{m-1}$. It follows that $|\theta_{m+1}| \leq C_0|\theta_m|^d + C_1|\theta_{m-1}| + C_2$. More loosely, we have

$$C_3 + |\theta_{m+1}| \leq (C_3 + |\theta_m|)^d (C_3 + |\theta_{m-1}|)^d. \tag{9}$$

Let us put $L_m(r) = \max_{|t|=r} |\theta_m(t)|$ and $u_m(r) = \log(C_3 + L_m(r))$. Note that $u_m(r)$ is monotonically increasing in r and tends to $+\infty$ as $r \to \infty$. It follows from (9) that $u_{m+1} \leq d(u_m + u_{m-1})$. Let $-\alpha$ and β be the roots of the quadratic equation $X^2 - dX - d = 0$ such that $0 < \alpha < 1$ and $\beta > d$. Then we have $u_{m+1} + \alpha u_m \leq \beta(u_m + u_{m-1})$. Since $0 < \alpha < 1$ and $M_m(r)$ is monotonically increasing, this estimate implies

$$\begin{aligned}
&\alpha\{\log M_{m+1}(r) + \log M_m(r)\} \\
&\quad \leq \log\{C_3 + M_{m+1}(|\lambda|^{m+1}r)\} + \alpha \log\{C_3 + M_m(|\lambda|^m r)\} \\
&\quad = u_{m+1}(r) + \alpha u_m(r) \\
&\quad \leq \beta^m\{u_1(r) + \alpha u_0(r)\} \\
&\quad \leq \beta^m\{\log(C_3 + M_1(|\lambda| r)) + \log(C_3 + M_0(r))\}.
\end{aligned}$$

Note that (8) implies $\eta_0(t) = \xi_1(\lambda t)$ and hence $N_0(r) = M_1(|\lambda| r)$. Hence the above estimate implies

$$\log M_{m+1}(r) + \log M_m(r) \leq C_4\{\log M_0(r) + \log N_0(r)\}$$

for $m = 0, \cdots, k-1$. Combining this estimate with Lemma 2, we obtain $\log M_0(r) + \log N_0(r) \geq C_5 \sum_{j=0}^{k-1} \log M_j(r) \geq C_6 r^\tau - C_7$. Lemma 3 is thus established.

7. Proof of Lemma 4. We put $F^k(x, y) = {}^t(f(x, y), g(x, y))$. In view of the form of the map $F: {}^t(x, y) \mapsto {}^t(y, ax + f(y))$, let us provide a weight d with the variable x and a weight 1 with the variable y. Then it is easy to see that $f(x, y)$ and $g(x, y)$ are homogeneous of order d^{k-1} and d^k with respect to these weights, respectively. Hence we have the following estimates:

$$\begin{aligned}
|f(x, y)| &\leq C_0\{1 + |x|^{1/d} + |x|\}^{dk-1}, \\
|g(x, y)| &\leq C_0\{1 + |x|^{1/d} + |y|\}^{dk}.
\end{aligned} \tag{10}$$

Since $(\xi(t), \eta(t))$ is an unstable curve, we have $\xi(\rho t) = f(\xi(t), \eta(t))$ and $\eta(\rho t) = g(\xi(t), \eta(t))$. Hence (10) implies

$$\begin{aligned}
M_0(|\rho| r) &\leq C_0\{1 + M_0(r)^{1/d} + N_0(r)\}^{dk-1}, \\
N_0(|\rho| r) &\leq C_0\{1 + M_0(r)^{1/d} + N_0(r)\}^{dk}.
\end{aligned} \tag{11}$$

Put $K(r) = \log M_0(r) + \log N_0(r)$. Then (11) implies

$$\begin{aligned} K(|\rho|r) &\leq (d^k+d^{k-1})\log\{1+M_0(r)^{1/k}+N_0(r)\}+C_1 \\ &\leq (d^k+d^{k-1})\{\log M_0(r)^{1/d}+\log N_0(r)\}+C_2 \\ &\leq (d^{k-1}+d^{k-2})K(r)+(d-1)(d^{k-1}+d^{k-2})\log N_0(r)+C_2 \\ &\leq (d^{k-1}+d^{k-2})K(r)+C_3\{\log N_0(r)+1\}. \end{aligned}$$

Let $\gamma=d^{k-1}+d^{k-2}$. Since $d\geq 2$, we have $1<\gamma<d^k$. Summarizing these estimates, we obtain

(12) $$K(|\rho|r)\leq \gamma K(r)+C_3\{\log N_0(r)+1\}, \qquad 1<\gamma<d^k.$$

Applying (12) repeatedly, we obtain

(13) $$\begin{aligned} K(|\rho|^m r) &\leq \gamma^m K(r)+C_3\sum_{n=0}^{m-1}\gamma^{m-n-1}\{\log N_0(|\rho|^n r)+1\} \\ &\leq \gamma^m K(r)+C_4(\gamma^m-1)\{\log N_0(|\rho|^m r)+1\}. \end{aligned}$$

On the other hand, Lemma 1 and Lemma 3 imply $K(r)\leq C_5 r^\tau+C_6$ and $K(|\rho|^m r)\geq C_7(|\rho|^m r)^\tau - C_8 = C_7(d^k)^m r^\tau - C_8$, respectively. Here we used the equality $|\rho|^\tau=d^k$ which follows from the definition of τ. Substituting these estimates into (13), we obtain

(14) $$C_4(\gamma^m-1)\{\log N_0(|\rho|^m r)+1\}\geq\{C_7(d^k)^m-C_5\gamma^m\}r^\tau-(C_8+\gamma^m C_6).$$

Since $d^k>\gamma$ (see (12)), there exists an $m\in \boldsymbol{N}$ such that $C_7(d^k)^m-C_5\gamma^m>0$. Choose and fix such an m. Then we have $\log N_0(|\rho|^m r)\geq C_9 r^\tau - C_{10}$. Replacing γ by $|\rho|^{-m}r$, we obtain

(15) $$\log N_0(r)\geq C_{11}r^\tau - C_{12}.$$

So far we have made the argument with the unstable curve $\Xi_0(t)$ and obtained the estimate (15). If we make the same argument with the unstable curve $\Xi_{-1}(t)$ instead of $\Xi_0(t)$, then we obtain an estimate for $N_{-1}(r)$ similar to (15). Notice that (8) implies $\xi_0(\lambda t)=\eta_{-1}(t)$ and hence $M_0(|\lambda|r)=N_{-1}(r)$. Thus we obtain

(16) $$\log M_0(r)\geq C_{13}r^\tau - C_{14}.$$

Estimates (15) and (16) establish Lemma 4.

As is noted in § 3, Main Theorem is an easy consequence of Lemma 1 and Lemma 4.

References

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