52. Centralizers of Galois Representations in Pro-l Pure Sphere Braid Groups

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The purpose of this note is to announce a result about exterior Galois representations in pro-l pure sphere braid groups by summarizing [9, 10]. Let $M_{0,n}$ be the moduli variety of the isomorphism classes of ordered n-pointed projective lines considered to be defined over a number field k, and let l be a rational odd prime. Then we have an exact sequence of profinite groups

$$(*)\qquad 1\longrightarrow \Gamma_0^{n,\text{pro-}l}\longrightarrow \pi_1^{(l)}(M_{0,n})\longrightarrow G_k\longrightarrow 1,$$

where

$$\begin{cases} G_k=\text{the absolute Galois group of } k,\\ \Gamma_0^{n,\text{pro-}l}=\text{the pro-}l\text{ completion of } \hat{\Gamma}_0^n:=\pi_1(M_{0,n}\otimes\bar{k}),\\ \pi_1^{(l)}(M_{0,n})=\pi_1(M_{0,n})/\ker(\hat{\Gamma}_0^n\longrightarrow\Gamma_0^{n,\text{pro-}l}).\end{cases}$$

Our main result can be stated as follows.

Theorem 1. *Let $\varphi_n: G_k\to \operatorname{Out}\Gamma_0^{n,\text{pro-}l}$ be the exterior Galois representation induced from the exact sequence (*). Then the centralizer of the Galois image $\varphi_n(G_k)$ in $\operatorname{Out}\Gamma_0^{n,\text{pro-}l}$ is isomorphic to S_3 when $n=4$, and to S_n when $n\geq 5$. Here S_n denotes the symmetric group of degree n.*

We say that an automorphism of $\pi_1^{(l)}(M_{0,n})$ is *Galois equivariant* if it induces identity on the quotient $G_k=\pi_1^{(l)}(M_{0,n})/\Gamma_0^{n,\text{pro-}l}$.

Theorem 2. *Let $E_k(M_{0,n})$ denote the quotient of the group of Galois equivariant automorphisms of $\pi_1^{(l)}(M_{0,n})$ modulo the inner automorphisms by $\Gamma_0^{n,\text{pro-}l}$, and let $\operatorname{Aut}_k M_{0,n}$ be the k-automorphism group of the variety $M_{0,n}$. Then the canonical homomorphism*

$$\Phi_n: \operatorname{Aut}_k M_{0,n}\longrightarrow E_k(M_{0,n})$$

gives a bijection.

As $\operatorname{Aut}_k M_{0,n}$ for $n\geq 5$ is known to be isomorphic to S_n [13], we see that Theorem 2 is a restatement of Theorem 1 by an argument of (profinite) group theory. Theorem 2 for $M_{0,4}=\boldsymbol{P}^1-\{0,1,\infty\}$ is proved in [8] as an application of Ihara's theory [4], Belyi's lifting [1], and a characterization of inertia groups in terms of nonabelian weight filtration [7]. For $n\geq 5$, it is shown by N. V. Ivanov [6] that the exterior automorphism group of the discrete group $\Gamma_0^n:=\pi_1(M_{0,n}(\boldsymbol{C}))$ is finite. Our theorem asserts that, while the pro-l completion $\Gamma_0^{n,\text{pro-}l}$ has infinite (nonabelian) exterior Galois symmetries by Ihara's result [5], the exterior *Galois equivariant* symmetries of it are again limited to being finite. This will support a conjecture about the arithmetic analogue of the Ivanov-McCarthy rigidity of

Teichmüller modular groups or conjectual etale-homotopical hyperbolicity of a "certain" class of varieties, inspired by Bogomolov [2], Grothendieck [3], Oda [11] or Parshin [12].

We briefly describe the lines of the proof. Let $\{a_1, \cdots, a_n\}$ be n distinct punctures on the sphere S^2, and let C_{ij} be the conjugacy class of $\Gamma_0^{n,\text{pro-}l}$ corresponding to the Dehn twists about simple closed curves which separates the pair $\{a_i, a_j\}$ from the others $(1\leq i,\ j\leq n)$. In [5], Y. Ihara called an automorphism of $\Gamma_0^{n,\text{pro-}l}$ preserving each of the C_{ij} $(1\leq i,\ j\leq n)$ *special*, and showed that a special automorphism of $\Gamma_0^{n,\text{pro-}l}$ which induces an inner automorphism on the quotient $\Gamma_0^{n-1,\text{pro-}l}$ is itself an inner automorphism. This and the following Key lemma enable us to reduce our theorem for $n\geq 5$ inductively to the case of $n=4$ established in [8].

Key lemma. *Every Galois equivariant automorphism f of $\pi_1^{(l)}(M_{0,n})$ preserves each conjugacy class C_{ij}, after composition with a suitable Galois equivariant automorphism coming from $S_n\cong \mathrm{Aut}_k M_{0,n}$.*

The proof of the Key lemma is achieved by applying the following three combinatorial lemmas. For $n\geq m\geq 3$ and a subset S of $\{1, \cdots, n\}$ with cardinality $n-m$, we have a homomorphism $p_S: \pi_1^{(l)}(M_{0,n})\to\pi_1^{(l)}(M_{0,m})$ obtained by forgetting the marked points on $\boldsymbol{P}^1$ corresponding to S. If S consists of a single element $\nu\in\{1, \cdots, n\}$, the kernel of p_ν $(=p_{\{\nu\}})$ is generated by the $C_{i\nu}$ $(1\leq i\leq n)$. In the following lemmas, let f be a given arbitrary Galois equivariant automorphism of $\pi_1^{(l)}(M_{0,n})$ and let $\mathfrak{X}_{ij}=\{x^a \mid x\in C_{ij}, a\in \boldsymbol{Z}_l\}$. Moreover we consider the situation (**): $p_S\circ f(\mathfrak{X}_{\lambda\mu})\neq 1$ and $p_\nu\circ p_S\circ f(\mathfrak{X}_{\lambda\mu})=1$, where $S\subset\{1, \cdots, n\}$, $\nu\in\{1, \cdots, m\}$ $(n\geq m\geq 4,\ n-m=|S|)$, and (λ, μ) is a given pair from $\{1, \cdots, n\}$ with $\lambda\neq\mu$.

Lemma 1. *Besides the hypothesis of (**), assume* $\bigcup_{1\leq i\leq n} f(\mathfrak{X}_{\lambda i})\subset \ker(p_\nu\circ p_S)$. *Then $p_S\circ f(\mathfrak{X}_{\lambda\mu})$ coincides with one of the $\mathfrak{X}_{\nu j}$ $(1\leq j\leq m, j\neq\nu)$.*

Lemma 2. *Under the hypothesis of (**), either* $\bigcup_{1\leq i\leq n} f(\mathfrak{X}_{\lambda i})$ *or* $\bigcup_{1\leq i\leq n} f(\mathfrak{X}_{\mu i})$ *is contained in* $\ker(p_\nu\circ p_S)$.

Lemma 3. *For each $\nu\in\{1, \cdots, n\}$, there exists at least one $\mathfrak{X}_{ij}$ $(1\leq i< j\leq n)$ such that $f(\mathfrak{X}_{ij})$ is contained in* $\ker(p_\nu)$.

A crucial point for the proof of Lemma 1 is that nonabelian weight filtration works properly in $\ker(p_\nu)$ under the given assumptions. Lemmas 2 and 3 are proved in this order by iterative applications of the preceding lemma(s).

These three lemmas have their graded Lie algebra versions for $n\geq m\geq 5$, and can be used to prove Galois rigidity of pure sphere braid Lie algebras for $n\geq 5$ formulated by P. Deligne.

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