48. An Ill-posed Estimate for a Class of Degenerate Quasilinear Elliptic Equations

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§ 1. **Introduction.** Let D be a domain in R^N, and let Γ be an open subset of ∂D, which is said to be an initial surface. We denote by O an origin in R^N. We suppose that O is the interior point of Γ. Let L be an elliptic operator in $\bar{D}$, which may be nonlinear. Let u be a solution of $L(u)=0$ in D. Then the ill-posed estimate in Cauchy's problem is the following: There are an open neighborhood U of O and two constants C, δ with $0<\delta<1$ such that

(1.1) $$\|U\|_{2,U\cap D}\leqq C(\|u\|_{1,\Gamma})^{\delta}(\|u\|_{3,D})^{1-\delta},$$

where $\|\ \|_i$ $(i=1,2,3)$ are some norms on Γ, $U\cap D$ and D, respectively. In particular, $\|\ \|_{1,\Gamma}$ means some quantity of initial data of u. The investigation with respect to ill-posed estimates of linear operators is referred to John's work [2]. The Hadamard's three circles theorem is close to the estimate (1.1). With respect to the nonlinear case, Výborný [7] has proved recently the Hadamard's three circles theorem for nonlinear uniformly elliptic operators.

The estimate (1.1) implies immediately the unique continuation property, which asserts that $u=0$ in $U\cap D$ if the initial data of u vanishes on Γ. For elliptic operators with linear principal parts the unique continuation property was extensively studied by many authors. Let $A(x, \xi)$ be a mapping from $D\times R^N$ into R^N such that for a.e. $x\in R^N$ and for all $\xi\in R^N$

$$|A(x, \xi)|\leqq C|\xi|^{p-1}, \qquad A(x, \xi)\cdot\xi\geqq c|\xi|^p$$

where c, $C>0$ and $p>1$. Then we consider particularly the elliptic operator L with

(1.2) $$L(u)=\operatorname{div}(A(x, \nabla u)\cdot\nabla u).$$

Recently, Martio [5] gave a counterexample of the form (1.2) such that the unique continuation property does not hold. In his counterexample, the function $A(x, \xi)$ and $u(x)$ are constructed skillfully under the conditions such as $p=N\geqq 3$, $D=\{x_N>0\}$ and $\Gamma=\{x_N=0\}$.

When $N=2$, the unique continuation property holds for the operators of (1.1) under some conditions (see e.g., [1] and [4]). However these method can not be applied to the case of $N\geqq 3$. The difficulty is originated from the degeneration of ellipticity. Thus there arises a question: If $N\geqq 3$, does the unique continuation property, moreover the ill-posed estimate hold for degenerate quasilinear elliptic operators?

In this paper we give a partial affirmative answer for the above question. We proceed along the line of [2] and [6], but we yield our estimate without using the Fourier transform.

§ 2. **Result.** We write $x=(x_1, \cdots, x_N)$, $x'=(x_1, \cdots, x_{N-1})$ and $y=x_N$. Thus $x=(x', y)$. In this paper we consider the operator

$$L_k(u)=\sum_{i=1}^{N} \partial_{x_i}((\partial_{x_i}u)^{2k+1}), \qquad k=0,1,2,\cdots,$$

which is a form of (1.2) and is a typical model of the degenerate quasilinear elliptic operator $\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i}u|^{p-2}\partial_{x_i}u)$ (see e.g., [3, Chap. 2]).

Let D and Γ be the same as in the previous section. We say that D is strictly convex at O, if there is a plane π passing through O, which meets $\bar{D}$ only at O. In this paper we impose the assumption

(A) Γ is of class C^1 and D is strictly convex at O. The positive y-axis is the ray perpendicular to π and $D\cap\{y<0\}=\varnothing$.

For $c>0$ we write

$$D_c=D\cap\{0<y<c\}, \qquad \Gamma_c=\Gamma\cap\{0<y<c\}.$$

From now on we fix a positive number a such that $a<1/2$ and $\partial D_a=\Gamma_a \cup (\bar{D}\cap\{y=a\})$.

Under the assumption (A) our aim is to prove

Theorem. *Let u be in $C^1(\bar{D}_a)$, and let its second derivatives be piecewise continuous in D_a. Let*

(2.1) $$|L_k(u)|\leqq K|u|^{2k+1} \qquad \text{in } D.$$

Then, if

$$\int_{D\cap\{y=a\}} (|u|+|\nabla u|)^{2k+2}dS\leqq M,$$

$$\int_{\Gamma_a} (|u|+|\nabla u|)^{2k+2}dS\leqq \varepsilon$$

and $\mu\varepsilon\leqq M$, it holds that

$$\int_{D_{a/2}} u^{2k+2}dx\leqq C\varepsilon^{a/2}M^{1-a/2},$$

where C and μ are positive constants depending only on k and K.

§ 3. **Lemmas.** First we prepare

Lemma 1. *For any nonnegative integer k, there is a positive constant c_k such that for $X, Y\in R$*

$$X[(X+Y)^{2k+1}-Y^{2k+1}]\geqq c_kX^{2k+2}, \qquad k=0,1,2,\cdots.$$

Proof. We set

$$f(t)=(1+t)^{2k+1}-t^{2k+1}, \qquad t\in R.$$

It is enough to prove that for $k\geqq 1$

(3.1) $$f(t)\geqq c_k.$$

When $t\geqq 0$, (3.1) is correct, since $f(0)=1$, $f(t)>0$ and $f(t)\to+\infty$ as $t\to+\infty$. When $t<0$, we use the equality $f(t)=f(-1-t)$ and we consider the two cases of $-1\leqq t<0$ and $t<-1$. Then (3.1) follows immediately. We complete the proof.

The following lemma is a slight modification of Poincaré's inequality. The proof is elementary, so we omit it.

Lemma 2. *Let $p\geqq 1$, and let u be in $C^1(\bar{D}_a)$. Then it holds that*

$$\int_{D_a}|u|^p dx \leqq C(p,a)\left[\int_{\Gamma_a}|u|^p dS+\int_{D_a}|\partial_y u|^p dx\right].$$

§ 4. **Proof of our theorem.** We denote by (,) the L^2-inner product in D_a. First we set $v(x',y)=\exp(\lambda y)\cdot u(x',y)$ for $\lambda<-1$.

By integration by parts we have

$$\begin{aligned}(4.1)\qquad & -(L_k(u), \exp((2k+1)\lambda y)\cdot\partial_y v)\\ &=\sum_{i=1}^{N-1}((\partial_{x_i}v)^{2k+1}, \partial_{x_i}\partial_y v)\\ &\quad+(\exp(-(2k+1)\lambda y)\cdot(\partial_y v-\lambda v)^{2k+1}, \partial_y(\exp((2k+1)\lambda y)\cdot\partial_y v))\\ &\quad-\sum_{i=1}^{N-1}\int_{\partial D_a}(\partial_{x_i}v)^{2k+1}\partial_y v\cdot\cos(x_i,\boldsymbol{n})dS\\ &\quad-\int_{\partial D_a}(\partial_y v-\lambda v)^{2k+1}\partial_y v\cdot\cos(y,\boldsymbol{n})dS,\end{aligned}$$

where $\boldsymbol{n}$ is an outernormal of ∂D_a and $(x_i,\boldsymbol{n})$, $(y,\boldsymbol{n})$ are the angles between x_i and $\boldsymbol{n}$, y and $\boldsymbol{n}$, respectively. On the other hand

$$((\partial_{x_i}v)^{2k+1}, \partial_{x_i}\partial_y v)=\frac{1}{2(k+1)}\int_{\partial D_a}(\partial_{x_i}v)^{2k+2}\cos(y,\boldsymbol{n})dS,$$

and the second term on the right-hand side of (4.1) equals

$$((\partial_y v-\lambda v)^{2k+1}, \partial_y^2 v+(2k+1)\lambda\partial_y v).$$

Thus (4.1) becomes

$$(4.2)\qquad -(L_k(u), \exp((2k+1)\lambda y)\cdot\partial_y v)=((\partial_y v-\lambda v)^{2k+1}, \partial_y^2 v+(2k+1)\lambda\partial_y v)+I_1,$$

where

$$\begin{aligned}I_1=&\frac{1}{2(k+1)}\sum_{i=1}^{N-1}\int_{\partial D_a}(\partial_{x_i}v)^{2k+2}\cos(y,\boldsymbol{n})dS\\ &-\sum_{i=1}^{N-1}\int_{\partial D_a}(\partial_{x_i}v)^{2k+1}\partial_y v\cdot\cos(x_i,\boldsymbol{n})dS\\ &-\int_{\partial D_a}(\partial_y v-\lambda v)^{2k+1}\partial_y v\cdot\cos(y,\boldsymbol{n})dS.\end{aligned}$$

Now we calculate the first term on the right-hand side of (4.2). First

$$((\partial_y v-\lambda v)^{2k+1}, \partial_y^2 v)=\sum_{j=0}^{2k+1}\binom{2k+1}{j}((\partial_y v)^j(-\lambda v)^{2k+1-j}, \partial_y^2 v).$$

Obviously

$$\begin{aligned}((\partial_y v)^j v^{2k+1-j}, \partial_y^2 v)=&-\frac{2k+1-j}{j+1}((\partial_y v)^{j+2}, v^{2k-j})\\ &+\frac{1}{j+1}\int_{\partial D_a}v^{2k+1-j}(\partial_y v)^{j+1}\cos(y,\boldsymbol{n})dS.\end{aligned}$$

Since $\binom{2k+1}{j}(2k+1-j)/(j+1)=\binom{2k+1}{j+1}$, we have

$$\begin{aligned}((\partial_y v-\lambda v)^{2k+1}, \partial_y^2 v)=&\lambda\sum_{j=-1}^{2k}\binom{2k+1}{j+1}(-\lambda)^{2k-j}\\ &\times((\partial_y v)^{j+2}, v^{2k-j})-\lambda(-\lambda)^{2k+1}(\partial_y v, v^{2k+1})+I_2,\end{aligned}$$

where

$$I_2=\sum_{j=0}^{2k+1}\binom{2k+1}{j}\frac{1}{j+1}\cdot\int_{\partial D_a}(-\lambda v)^{2k+1-j}(\partial_y v)^{j+1}\cos(y,\boldsymbol{n})dS.$$

It becomes

$$((\partial_y v-\lambda v)^{2k+1},\partial_y^2 v)=\lambda\sum_{j=0}^{2k+1}\binom{2k+1}{j}((\partial_y v)^{j+1},(-\lambda v)^{2k+1-j})$$
$$+\lambda^{2k+2}(\partial_y v, v^{2k+1})+I_2.$$

And we have

$$((\partial_y v-\lambda v)^{2k+1},\partial_y v)=\sum_{j=0}^{2k+1}\binom{2k+1}{j}((\partial_y v)^{j+1},(-\lambda v)^{2k+1-j}).$$

From the above inequalities it follows that

$$\begin{aligned}&\text{the right-hand side of (4.1)}=2(k+1)\lambda\\ &\times\left[\sum_{j=0}^{2k+1}\binom{2k+1}{j}((\partial_y v)^{j+1},(-\lambda v)^{2k+1-j})-(\partial_y v,(-\lambda v)^{2k+1})\right]\\ &-(2k+1)\lambda^{2k+2}(\partial_y v, v^{2k+1})+I_1+I_2\\ &=2(k+1)\lambda(\partial_y v,(\partial_y v-\lambda v)^{2k+1}-(-\lambda v)^{2k+1})+\sum_{j=1}^{3}I_j,\end{aligned}$$

where

$$I_3=-\frac{2k+1}{2(k+1)}\lambda^{2k+2}\int_{\partial D_a}v^{2k+2}\cos(y,\boldsymbol{n})dS.$$

Combining this and (4.1) with Lemma 1, we conclude that

$$(L_k(u),\exp((2k+1)\lambda y)\cdot\partial_y v)\geqq 2(k+1)c_k|\lambda|(1,(\partial_y v)^{2k+2})-\sum_{j=1}^{3}I_j. \tag{4.3}$$

By Cauchy's inequality

$$|(L_k(u),\exp((2k+1)\lambda y)\cdot\partial_y v)|\leqq\frac{1}{2(k+1)}\int_{D_a}(\partial_y v)^{2k+2}dx$$
$$+\frac{2k+1}{2(k+1)}\int_{D_a}\exp(2(k+1)\lambda y)\cdot|L_k(u)|^{2(k+1)/(2k+1)}dx.$$

Further we easily see that

$$\left|\sum_{j=1}^{3}I_j\right|\leqq C\int_{\partial D_a}(|\nabla v|^{2k+2}+|\lambda|^{2k+2}v^{2k+2})dS,$$

where C depends only on k. Combining these inequalities with (4.3) and (2.1) we have

$$\int_{D_a}(\partial_y v)^{2k+2}dx\leqq C|\lambda|^{-1}\left[\int_{D_a}v^{2k+2}+\int_{\partial D_a}(|\nabla v|^{2k+2}+|\lambda|^{2k+2}v^{2k+2})dS\right]$$

for $\lambda<-\lambda_0$ ($\lambda_0>0$). Applying Lemma 2 for $p=2k+2$, we obtain

$$\int_{D_{a/2}}v^{2k+2}dx\leqq C|\lambda|^{2k+2}\left[\int_{\Gamma_a}(|u|+|\nabla u|)^{2k+2}dS\right.$$
$$\left.+\exp(2(k+1)\lambda a)\int_{D\cap\{y=a\}}(|u|+|\nabla u|)^{2k+2}dS.\right]$$

Hence

$$\int_{D_{a/2}}u^{2k+2}dx\leqq C|\lambda|^{2k+2}\exp(-(k+1)\lambda a)\cdot(\varepsilon+M\exp(2(k+1)\lambda a)).$$

Taking λ_0 as large as desired, we note that

$$|\lambda|^{2k+2}\exp(-(k+1)\lambda a)\leqq C\exp(-3(k+1)\lambda a/2).$$

Setting $\lambda=-1/(k+1)\log(M/\varepsilon)$, we obtain finally

$$\int_{D_{a/2}} u^{2k+2}dx \leqq C(\varepsilon^{1-(3a/2)}M^{3a/2}+\varepsilon^{a/2}M^{1-(a/2)}).$$

This completes the proof.

References

[1] Alessandrini, G.: Critical points of solutions to the p-Laplace equation in dimension two. Boll. Un. Mat. Ital., (7), **1**-A, 239–246 (1987).
[2] John, F.: Continuous dependence on data for solutions of partial differential equations with a prescribed bound. Comm. Pure Appl. Math., **13**, 551–585 (1960).
[3] Lions, J. L.: Quelques méthodes de résolutions des problèmes aux limites non linéaires. Dunod/Gauthier Villars (1969).
[4] Manfredi, J. J.: p-harmonic functions in the plane. Proc. Amer. Math. Soc., **103**, 473–479 (1988).
[5] Martio, O.: Counterexamples for unique continuation. Manuscripta Math., **60**, 21–47 (1988).
[6] Nirenberg, L.: Uniqueness in Cauchy problems for differential equations with constant leading coefficients. Comm. Pure Appl. Math., **10**, 89–105 (1957).
[7] Výborný, R.: The Hadamard three-circles theorems for nonlinear equations. J. Austral. Math. Soc., A **49**, 297–302 (1990).