45. A Note on Wada's Group Invariants of Links

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In [3], Wada investigated group invariants of links derived from representations of the n-string braid group B_n to the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n of rank n. He classified "shift type representations" through computer experiment, and interpreted the group invariants of links derived from these representations in terms of the link groups with one exception. The exceptional representation $\gamma: B_n \to \mathrm{Aut}(F_n)$ is given as follows (see [3, § 5]).

$$\begin{aligned} x_i\gamma(\sigma_i) &= x_i^2 x_{i+1}, \\ x_{i+1}\gamma(\sigma_i) &= \bar{x}_{i+1}\bar{x}_i x_{i+1}, \\ x_j\gamma(\sigma_i) &= x_j \qquad (j \neq i, i+1). \end{aligned} \tag{1}$$

Here $\{\sigma_1, \cdots, \sigma_{n-1}\}$ is the standard generator system of B_n, and $\{x_1, \cdots, x_n\}$ is a free basis of F_n. Wada's group invariant $G_\gamma(L)$ of a link L associated with γ is defined as follows: Let b be an element of B_n such that the closed braid obtained from b is isotopic to L. Then,

$$G_\gamma(L) = \langle x_1, \cdots, x_n \mid x_i\gamma(b) = x_i \ (1 \leq i \leq n)\rangle. \tag{2}$$

The purpose of this note is to prove the following theorem, which answers the question of Wada in [3, § 5].

Theorem. *$G_\gamma(L) \cong Z * \pi_1(\Sigma_2(L))$, where $\Sigma_2(L)$ is the 2-fold covering of S^3 branched over L.*

Proof of theorem. Let B_{n+1} be the $(n+1)$-string braid group, and let σ_i $(0 \leq i \leq n-1)$ be the element of B_{n+1} as shown in Figure 1.

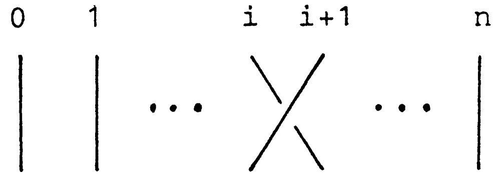

Fig. 1

We sometimes consider B_n as the subgroup of B_{n+1} generated by $\{\sigma_1, \cdots, \sigma_{n-1}\}$. Let $F_{n+1} = \langle x_0, x_1, \cdots, x_n \rangle$ be the free group of rank $n+1$, and let $\gamma: B_{n+1} \to \mathrm{Aut}(F_{n+1})$ be the representation defined by (1). Put $a_i = x_{i-1}x_i \in F_{n+1}$ $(1 \leq i \leq n)$. Then $\{x_0, a_1, \cdots, a_n\}$ is a free basis of F_{n+1}, and for each $\sigma_i \in B_n$ $(1 \leq i \leq n-1)$, we have $x_0\gamma(\sigma_i) = x_0$, and

$$\begin{aligned} a_i\gamma(\sigma_i) &= a_i a_{i+1}, \\ a_{i+2}\gamma(\sigma_i) &= \bar{a}_{i+1} a_{i+2}, \\ a_j\gamma(\sigma_i) &= a_j \qquad (j \neq i, i+2). \end{aligned} \tag{3}$$

Let L' be the $(n+1)$-string closed braid obtained from $b \in B_n \subset B_{n+1}$. Since the subgroups $\langle x_0 \rangle$ and $\langle a_1, \cdots, a_n \rangle$ are $\gamma(b)$-invariant, we have

$$G_\gamma(L') \cong \langle x_0, x_1, \cdots, x_n \mid x_i \gamma(b) = x_i \ (0 \leq i \leq n) \rangle$$
$$\cong \langle x_0 \rangle * \langle a_1, \cdots, a_n \mid a_i \gamma(b) = a_i \ (1 \leq i \leq n) \rangle.$$

Similarly, we see

$$G_\gamma(L') \cong \langle x_0 \rangle * G_\gamma(L).$$

Hence we have

(4) $$G_\gamma(L) \cong \langle a_1, \cdots, a_n \mid a_i \gamma(b) = a_i \ (1 \leq i \leq n) \rangle.$$

On the other hand, since L' is the split sum of L and a trivial knot, we see

(5) $$\pi_1(\Sigma_2(L')) \cong Z * \pi_1(\Sigma_2(L)).$$

In the following, we prove that $\pi_1(\Sigma_2(L'))$ is isomorphic to $G_\gamma(L)$. Let $\mathcal{O}(L')$ be the π-orbifold group of L'; that is, the quotient group of $\pi_1(S^3 - L')$ by the normal subgroup generated by the squares of the meridians of L' (cf. [2]). Then $\mathcal{O}(L')$ is a split extension of Z_2 by $\pi_1(\Sigma_2(L'))$. Let $G_{n+1} = \langle x_0, x_1, \cdots, x_n \mid x_i^2 = 1 \ (0 \leq i \leq n) \rangle$, and let $\rho: B_{n+1} \to \mathrm{Aut}(G_{n+1})$ be the representation obtained by deducting the Artin representation (see [1, p. 25], [3, § 1]); that is,

$$x_i \rho(\sigma_i) = x_{i+1}, \quad x_{i+1}\rho(\sigma_i) = x_{i+1} x_i x_{i+1}, \quad x_j \rho(\sigma_i) = x_j \quad (j \neq i, \ i+1).$$

(Here each generator σ_i is taken to be the inverse of that in [1].) Since the link group is obtained from the Artin representation, the π-orbifold group is given by

$$\mathcal{O}(L') \cong G_{n+1} / \langle x_i \rho(b) = x_i \ (0 \leq i \leq n) \rangle.$$

Put $a_i = x_{i-1} x_i \in G_{n+1}$ $(1 \leq i \leq n)$, and let τ be the automorphism of the free subgroup $\langle a_1, \cdots, a_n \rangle$ defined by

$$a_i \tau = (a_1 \cdots a_{i-1}) \bar{a}_i \overline{(a_1 \cdots a_{i-1})}.$$

Then we see

$$G_{n+1} \cong \langle x_0, a_1, \cdots, a_n \mid x_0^2 = 1, \ x_0 a_i \bar{x}_0 = a_i \tau \ (1 \leq i \leq n) \rangle.$$

Since $x_0 \rho(b) = x_0$ and $\langle a_1, \cdots, a_n \rangle$ is $\rho(b)$-invariant, we have

$$\mathcal{O}(L') \cong \langle x_0, a_1, \cdots, a_n \mid x_0^2 = 1, \ x_0 a_i \bar{x}_0 = a_i \tau, \ a_i \rho(b) = a_i \ (1 \leq i \leq n) \rangle.$$

For each $\sigma_i \in B_n$ $(1 \leq i \leq n)$, we see the restriction of $\rho(\sigma_i)$ to the free subgroup $\langle a_1, \cdots, a_n \rangle$ is equal to the restriction of $\gamma(\sigma_i)$ to the subgroup $\langle a_1, \cdots, a_n \rangle$ given by (3). Now, let $H(L')$ be the group defined by

$$H(L') \cong \langle a_1, \cdots, a_n \mid a_i \rho(b) = a_i \ (1 \leq i \leq n) \rangle.$$

Then $H(L')$ is isomorphic to $G_\gamma(L)$ by (4). Further, τ induces an automorphism of $H(L')$, since $\rho(b)\tau = \tau\rho(b)$. Hence $H(L')$ is naturally isomorphic to the normal subgroup $\mathcal{O}(L')$ of index 2, and therefore $G_\gamma(L) \cong H(L') \cong \pi_1(\Sigma_2(L'))$. This completes the proof by (5).

References

[1] J. S. Birman: Braids, links, and mapping class groups. Ann. Math. Studies, **82**, Princeton Univ. Press and Univ. of Tokyo Press (1974).

[2] M. Boileau and B. Zimmermann: The π-orbifold group of a link. Math. Z., **200**, 187–208 (1989).

[3] M. Wada: Group invariants of links (to appear in Topology).