44. On the Existence and Regularity of the Solution of Stokes Problem in Arbitrary Dimension

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(Communicated by Kunihiko KODAIRA, M. J. A., May 13, 1991)

Let Ω be a bounded and connected open set of $\boldsymbol{R}^N$ ($N\geq 2$) and let f, ϕ, g be three given functions that satisfy the compatibility condition: $\int_\Omega \phi(x)\,dx = \int_\Gamma g\cdot n\, d\sigma$, where n denotes the exterior normal to Γ. Recall the Stokes problem with unit viscosity:

Find a pair $(u.p)$ *such that*:

(1)–(3) $\quad -\Delta u+\nabla p=f$ in Ω, $\quad \operatorname{div} u=\phi$ in Ω, $\quad u=g$ on Γ.

The homogeneous case corresponds to $\phi=0$ and $g=0$.

Let $m\geq -1$ be an integer; let r denote an arbitrary real number such that $1<r<\infty$ and let r' be its conjugate: $1/r+1/r'=1$. We agree to denote by $\boldsymbol{X}$ the product space X^N. This note establishes that, for each f in $\boldsymbol{W}^{m,r}(\Omega)$, ϕ in $W^{m+1,r}(\Omega)$ and g in $\boldsymbol{W}^{m+2-1/r,r}(\Gamma)$, problem (1)–(3) has a unique solution u in $\boldsymbol{W}^{m+2,r}(\Omega)$ and p in $W^{m+1,r}(\Omega)/\boldsymbol{R}$ that depends continuously upon the data. The regularity hypotheses that we impose on the boundary are optimal when $m\geq 0$. For a smoother boundary, these results are of course not new (cf. Cattabriga [6] and Geymonat [7]), but the proof that we present here is new.

Our proof is based on the following fundamental orthogonal decomposition, which is very closely related to the Stokes problem:

$$\boldsymbol{W}^{m+2,r}(\Omega)\cap \boldsymbol{W}_0^{1,r}(\Omega)=(\boldsymbol{W}^{m+2,r}(\Omega)\cap V_{1,r})\oplus(\boldsymbol{W}^{m+2,r}(\Omega)\cap G_{1,r}), \tag{4}$$

where

$$V_{1,r}=\{v\in \boldsymbol{W}_0^{1,r}(\Omega)\,;\ \operatorname{div} v=0\} \text{ and } G_{1,r}=\{v\in \boldsymbol{W}_0^{1,r}(\Omega)\,;\ -\Delta v=\nabla q,\ q\in L_0^r(\Omega)\}.$$

First, for $m\geq 0$, we shall establish (4) by showing that the homogeneous Stokes problem is elliptic in the sense of Agmon-Douglis-Nirenberg [2]; this will immediately yield the desired result for such m. Unfortunately, the material in [2] does not apply when $m=-1$. We shall instead, solve by duality a weaker problem (an approach already used by Giga [8]), and then complete by interpolation our desired result for $m=-1$.

Proposition 1. *Let $m\in N$ and let the domain Ω be $C^{m+1,1}$. Assume that the homogeneous Stokes problem has a solution $u\in \boldsymbol{W}^{2,r}(\Omega)$ and $p\in W^{1,r}(\Omega)$. If in addition $f\in \boldsymbol{W}^{m,r}(\Omega)$, then $u\in \boldsymbol{W}^{m+2,r}(\Omega)$, $p\in W^{m+1,r}(\Omega)$ and*

$$\|u\|_{\boldsymbol{W}^{m+2,r}(\Omega)}+\|p\|_{W^{m+1,r}(\Omega)/\boldsymbol{R}}\leq C\|f\|_{\boldsymbol{W}^{m,r}(\Omega)}. \tag{5}$$

Proof. Following the proof of Proposition 2.2 in Temam [10], p. 33, we show that the homogeneous Stokes problem is an elliptic system in the

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sense of Agmon-Douglis-Nirenberg [2] (pp. 38–39 and 42–43). When Ω is C^{m+2}, Theorem 10.5, p. 78 [2], yields a weaker estimate than (5), namely:

$$\|u\|_{W^{m+2,r}(\Omega)}+\|p\|_{W^{m+1,r}(\Omega)/R}\leq C_1(\|f\|_{W^{m,r}(\Omega)}+d_r\|u\|_{L^r(\Omega)}).$$

But, since the domain is bounded and the solution is unique, according to Remark 2 pp. 668–669 in [1], we can take $d_r=0$. Furthermore, applying the material of Grisvard [10], the estimate of Theorem 10.5 is also valid for $C^{m+1,1}$ domains.

Proposition 1 is not an existence result, but it permits to obtain a regularity result whenever a solution exists. With this proposition, we can prove (4).

Proposition 2. *Let $m\in \boldsymbol{N}$ and let Ω be $C^{m+1,1}$. Then the decomposition* (4) *holds.*

Proof. Let E denote the space in the left-hand side of (4) and F the space in the right-hand side. To prove the equality, let us establish that F is closed and dense in E.

As far as the closure is concerned, let u_n be a sequence of F that converges to some element u in $W^{m+2,r}(\Omega)\cap W_0^{1,r}(\Omega)$. Then u_n has the form $u_n=v_n+w_n$ with $v_n\in W^{m+2,r}(\Omega)\cap V_{1,r}$ and $w_n\in W^{m+2,r}(\Omega)\cap G_{1,r}$. Furthermore, by definition, w_n satisfies: $-\Delta w_n=\nabla q_n$, where $q_n\in W^{m+1,r}(\Omega)$. Set $f_n=-\Delta u_n=-\Delta v_n+\nabla q_n$. Thus, the pair $(v_n, q_n)\in W^{m+2,r}(\Omega)\times W^{m+1,r}(\Omega)$ is the solution of a homogeneous Stokes problem with right-hand side f_n. Therefore, since f_n tends to $-\Delta u$ in $W^{m,r}(\Omega)$, Proposition 1 implies that the sequences v_n, q_n are bounded respectively in $W^{m+2,r}(\Omega)\subset W_0^{1,r}(\Omega)$ and $W^{m+1,r}(\Omega)/\boldsymbol{R}$. As a consequence, w_n is also bounded in $W^{m+2,r}(\Omega)\cap W_0^{1,r}(\Omega)$ and both $v_n\rightharpoonup v$ and $w_n\rightharpoonup w$ weakly in $W^{m+2,r}(\Omega)\cap W_0^{1,r}(\Omega)$, with $v\in V_{1,r}$ and $w\in G_{1,r}$.

To prove the density, let L be an element of E' that vanishes on F and let us show that $L=0$. It is easy to check that L has a unique extension $\tilde{L}\in W^{-1,r'}(\Omega)$ that vanishes on $V_{1,r}\oplus G_{1,r}$. Applying a simplified version (cf. for instance [4]) of de Rham's theorem, this implies in particular that $\tilde{L}=\nabla q$ for some $q\in L^{r'}(\Omega)$. Then, introducing the solution $z\in W_0^{1,r'}(\Omega)$ of the problem $-\Delta z=\nabla q$, we have for all $w\in G_{1,r}$: $\langle z, -\Delta w\rangle=0$; i.e. $\langle z, \nabla p\rangle=0$ for all $p\in L^r(\Omega)$. This means that $\operatorname{div} z=0$; hence $z=0$ and in turn $\tilde{L}=0$.

With this, the homogeneous Stokes problem reduces to a Laplace equation.

Theorem 3. *Let $m\in \boldsymbol{N}$ and let Ω be $C^{m+1,1}$. For each $f\in W^{m,r}(\Omega)$, the homogeneous Stokes problem has a unique solution $u\in W^{m+2,r}(\Omega)$ and $p\in W^{m+1,r}(\Omega)/\boldsymbol{R}$ and,*

$$\|u\|_{W^{m+2,r}(\Omega)}+\|p\|_{W^{m+1,r}(\Omega)/R}\leq C\|f\|_{W^{m,r}(\Omega)}. \tag{6}$$

Proof. Let $v\in W_0^{1,r}(\Omega)$ be the solution of the problem $-\Delta v=f$ in Ω. It stems from the well-known regularity properties of this operator that $v\in W^{m+2,r}(\Omega)$ with continuous dependence on f. Then, by virtue of Proposition 2, $v=u+w$, with $u\in W^{m+2,r}(\Omega)\cap V_{1,r}$ and $w\in W^{m+2,r}(\Omega)\cap G_{1,r}$. But since $-\Delta w=\nabla p$, with $p\in W^{m+1,r}(\Omega)$, we derive immediately that the pair u and p satisfies: $-\Delta u+\nabla p=f$, $\operatorname{div} u=0$.

Then the non homogeneous problem follows by using an adequate lifting operator.

Theorem 4. *Let $m \in \boldsymbol{N}$ and let Ω be $C^{m+1,1}$. For each $f \in \boldsymbol{W}^{m,r}(\Omega)$, $\phi \in W^{m+1,r}(\Omega)$ and $g \in \boldsymbol{W}^{m+2-1/r,r}(\Gamma)$ such that: $\int_\Omega \phi(x)dx = \int_\Gamma g \cdot n d\sigma$, the non homogeneous Stokes problem* (1)–(3) *has a unique solution $u \in \boldsymbol{W}^{m+2,r}(\Omega)$ and $p \in W^{m+1,r}(\Omega)/\boldsymbol{R}$ and,*

(7) $\|u\|_{W^{m+2,r}(\Omega)} + \|p\|_{W^{m+1,r}(\Omega)/R} \le C(\|f\|_{W^{m,r}(\Omega)} + \|\phi\|_{W^{m+1,r}(\Omega)} + \|g\|_{W^{m+2-1/r,r}(\Gamma)})$.

Proof. Owing to the compatibility condition, (cf. [3] or [4]), we can associate with ϕ and g a function $u_0 \in \boldsymbol{W}^{m+2,r}(\Omega)$, such that div $u_0 = \phi$ in Ω, $u_0 = g$ on Γ and

(8) $$\|u_0\|_{W^{m+2,r}(\Omega)} \le C(\|\phi\|_{W^{m+1,r}(\Omega)} + \|g\|_{W^{m+2-1/r,r}(\Gamma)}).$$

This reduces the problem to a homogeneous one and the result follows from Theorem 3.

There is no proof of the a priori estimates of Proposition 1 for $m = -1$ and so we cannot handle this case directly. But we can solve by duality a simplified version of the Stokes problem that corresponds to $m = -2$. The following proposition is party due to Giga [8].

Proposition 5. *Let Ω be $C^{1,1}$. For each $g \in \boldsymbol{W}^{-1/r,r}(\Gamma)$ satisfying $g \cdot n = 0$, the problem*

(9)–(11) $\quad -\Delta v + \nabla q = 0$ *in* Ω, $\quad \operatorname{div} v = 0$ *in* Ω, $\quad v = g$ *on* Γ,

has a unique solution $v \in \boldsymbol{L}^r(\Omega)$ and $q \in W^{-1,r}(\Omega)/\boldsymbol{R}$ and,

(12) $$\|v\|_{L^r(\Omega)} + \|q\|_{W^{-1,r}(\Omega)/R} \le C\|g\|_{W^{-1/r,r}(\Gamma)}.$$

Proof. First let us show that if $v \in \boldsymbol{L}^r(\Omega)$ satisfies (9) and (10) then the trace of v on Γ belongs to $\boldsymbol{W}^{-1/r,r}(\Gamma)$, so that the bounday condition (11) makes sense. On one hand, the fact that div $v = 0$ implies that $v \cdot n \in W^{-1/r,r}(\Gamma)$. On the other hand, it follows from (9) and (10) that $v \in T$, where $T = \{w \in \boldsymbol{L}^r(\Omega);\ \Delta w \in X',\ \operatorname{div} w = 0\}$ and $X = \{w \in \boldsymbol{W}_0^{1,r'}(\Omega);\ \operatorname{div} w \in W_0^{1,r'}(\Omega)\}$. Then a density argument yields the Green's formula:

(13) $$\left\langle v, \frac{\partial \phi}{\partial n} \right\rangle = \langle v, \Delta\phi \rangle - \langle \Delta v, \phi \rangle, \quad \forall \phi \in Y, \quad \forall v \in T,$$

where $Y = \boldsymbol{W}^{2,r'}(\Omega) \cap X$. By observing that the range space of the operator $\partial/\partial n$ on Y is the space $Z = \{w \in \boldsymbol{W}^{1/r,r'}(\Gamma);\ w \cdot n = 0\}$, and that its dual space Z' has the identification: $Z' = \{g \in \boldsymbol{W}^{-1/r,r}(\Gamma);\ g \cdot n = 0\}$, we see that, by virtue of (13), the tangential trace of v belongs precisely to Z'.

Now, the proof is based on a duality argument developed by Lions-Magenes [11]. Applying (13), we readily derive that problem (9)–(11) has the equivalent variational formulation (that can also be found in [8]): *find $v \in \boldsymbol{L}^r(\Omega)$ and $q \in W^{-1,r}(\Omega)/\boldsymbol{R}$ such that*:

$$\int_\Omega v(-\Delta u + \nabla p)dx - \langle q, \operatorname{div} u \rangle = \left\langle g, \frac{\partial u}{\partial n} \right\rangle_\Gamma \quad \forall u \in Y, \quad \forall p \in W^{1,r'}(\Omega).$$

But, owing to Theorem 4, for each $f \in \boldsymbol{L}^{r'}(\Omega)$ and $\phi \in W_0^{1,r'}(\Omega) \cap L_0^{r'}(\Omega)$ (where $L_0^{r'}(\Omega)$ denotes the subspace of functions of $L^{r'}(\Omega)$ with zero mean value), there exists a unique solution $u \in Y$ and $p \in W^{1,r'}(\Omega)/\boldsymbol{R}$ of the problem:

$$-\Delta u+\nabla p=f \text{ in } \Omega, \quad \operatorname{div} u=\phi \text{ in } \Omega, \quad u=0 \text{ on } \Gamma.$$

Moreover, it follows from the continuity of the mapping $\partial/\partial n: W^{2,r'}(\Omega) \to W^{1/r,r'}(\Gamma)$ and the estimate (7), that the mapping $(f,\phi)\mapsto\langle g, \partial u/\partial n\rangle_\Gamma$ defines an element of the dual space of $L^{r'}(\Omega)\times[W_0^{1,r'}(\Omega)\cap L_0^{r'}(\Omega)]$, with norm bounded by $C\|g\|_{W^{-1/r,r}(\Gamma)}$. Then the theorem follows from Riesz' representation theorem.

The next corollary, which relaxes the compatibility condition on the data g, is an easy consequence of Proposition 5.

Corollary 6. *Proposition 5 is also valid for each $g\in W^{-1/r,r}(\Gamma)$ satisfying $\langle g\cdot n, 1\rangle=0$.*

Then by interpolating between the results of Corollary 6 and Theorem 4 with $m=0$, $f=0$ and $\phi=0$, we obtain:

Corollary 7. *Let Ω be $C^{1,1}$. For each $g\in W^{1-1/r,r}(\Gamma)$ satisfying $\int_\Gamma g\cdot n d\sigma=0$, the solution of problem (9)–(11) has the regularity: $v\in W^{1,r}(\Omega)$ and $q\in L^r(\Omega)/\boldsymbol{R}$ and,*

$$\|v\|_{W^{1,r}(\Omega)}+\|q\|_{L^r(\Omega)/\boldsymbol{R}}\le C\|g\|_{W^{1-1/r,r}(\Gamma)}.$$

From this and the isomorphism properties of the divergence operator, we derive:

Corollary 8. *Let Ω be $C^{1,1}$. For each $\phi\in L_0^r(\Omega)$, problem (1)–(3), with $f=0$ and $g=0$, has a unique solution $u\in W_0^{1,r}(\Omega)$ and $p\in L^r(\Omega)/\boldsymbol{R}$ and,*

$$\|u\|_{W^{1,r}(\Omega)}+\|p\|_{L^r(\Omega)/\boldsymbol{R}}\le C\|\phi\|_{L^r(\Omega)}. \tag{14}$$

This last corollary permits to complete the statement of Proposition 2; it gives the analogue of the well-known decomposition: $H_0^1(\Omega)=V\oplus V^\perp$ (cf. for instance [9]).

Proposition 9. *If Ω is $C^{1,1}$, then $W_0^{1,r}(\Omega)=V_{1,r}\oplus G_{1,r}$.*

Finally, applying the same arguments as in Theorems 3 and 4, we easily derive:

Theorem 10. *Let Ω be $C^{1,1}$. For each $f\in W^{-1,r}(\Omega)$, $\phi\in L^r(\Omega)$ and $g\in W^{1-1/r,r}(\Gamma)$ satisfying: $\int_\Omega \phi(x)dx=\int_\Gamma g\cdot n d\sigma$, problem (1)–(3) has a unique solution $u\in W^{1,r}(\Omega)$ and $p\in L^r(\Omega)/\boldsymbol{R}$ and,*

$$\|u\|_{W^{1,r}(\Omega)}+\|p\|_{L^r(\Omega)/\boldsymbol{R}}\le C\{\|f\|_{W^{-1,r}(\Omega)}+\|\phi\|_{L^r(\Omega)}+\|g\|_{W^{1-1/r,r}(\Gamma)}\}.$$

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