40. *Merger of Chaotic Bands in Period-doubling Cascades*

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1. Introduction. One-dimentional recursion relation $x_{n+1}=F(x_n, \lambda)$, where $F(x, \lambda)$ is a real-valued function of real variables x and λ, which is smooth in λ, provided an important example as to the nature of the onset of chaos in dissipative dynamical systems. There has been much effort devoted to describing the properties of these maps. The sequence of period doubling bifurcations and its universal scaling behaviour have received particularly intense investigation.

Feigenbaum studied the quadratic one-dimentional maps (cf. [2])

$$f_\lambda(x)=\lambda x(1-x), \qquad 1\leq\lambda\leq 4.$$

When $\lambda<3$, successive iterations $x_{n+1}=\lambda x_n(1-x_n)$ converge to a point attractor of period one. As λ is increased up to the critical parameter value λ_∞ ($\sim$3.5700), known as Feigenbaum point, one obtains period-doubling bifurcations leading to period-2^n orbits. It is not known whether Feigenbaum point λ_∞ is rational or irrational number. See Fig. 1. When one

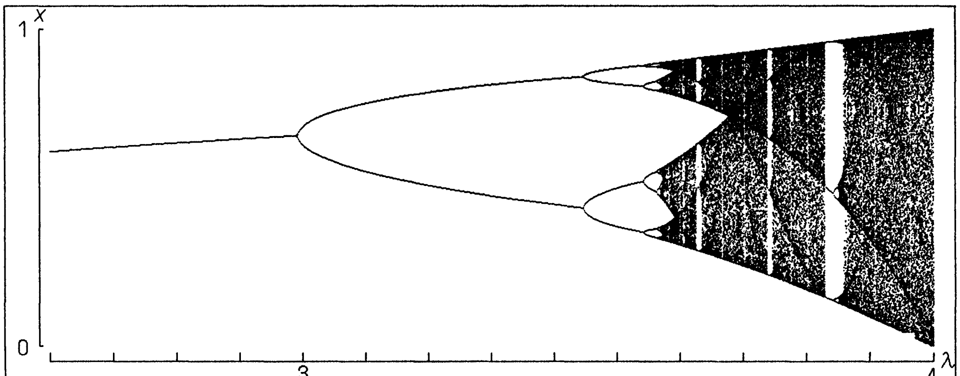

Fig. 1. Bifurcation diagram for the system of equation $f_\lambda(x)=\lambda x(1-x)$.

Fig. 1. Merger of chaotic bands.

periodic orbit turns to be unstable, as we increase λ, another stable orbit is created, with twice the period of the first one. The new orbit itself turns to be unstable at a larger value of λ and another stable orbit is created, with four times the period of the first one, etc. Thus a period-2^k orbit bifurcates, at λ_k, from a period-2^{k-1} orbit. The resulting 'Feigenbaum Sequence' $\{\lambda_k\}$, converges increasingly to the critical value λ_∞, with

$$\lim_{k\to\infty}\frac{\lambda_k-\lambda_{k-1}}{\lambda_{k+1}-\lambda_k}=\delta_F.$$

Hence this sequence has asymptotic behaviour for suitable constant c:

$$\lambda_\infty-\lambda_k\sim c\cdot\delta_F^{-k}$$

$\delta_F=4.66920\cdots$ is called 'Feigenbaum constant'. See [1].

As $\lambda>\lambda_\infty$, the chaotic bands and the 'reverse bifurcation of the chaotic bands' appear. Namely the 2^k bands are made of the merger of 2^{k+1} bands. When λ decreases from the parameter value λ_{M_0} ($\sim 3.6785\cdots$) to λ_∞, one chaotic band splits into two bands at λ_{M_0}, and two chaotic bands split into four bands at λ_{M_1} ($\sim 3.59257\cdots$), etc. Moreover, the resulting sequence $\{\lambda_{M_i}\}$ converges decreasingly to the critical value λ_∞, with the same asymptotic behaviour for suitable constant c':

$$\lambda_{M_k}-\lambda_\infty\sim c'\cdot\delta_F^{-k}.$$

The important discovery was the recognition that this behaviour is universal, and holds for all maps of the interval

$$x_{n+1}=F(x_n,\lambda),$$

where F has a single quadratic maximum, and a negative Schwartzian derivative.

The aim of the present work is to give the precise values of the parameters λ_{M_0} and λ_{M_1} as functions of λ.

The details are contained in [7].

2. Bands merging parameters. In this section we consider the process of bands merging and in particular, the situation where the first and the second bands merging begin.

Definition (Li-Yorke chaos [4]). Let I be an interval and let $f: I\to I$ be continuous. f is chaotic if there is an uncountable set $S\subset I$ (containing no periodic points), which satisfies the following conditions:

1. $\forall p, q\in S$ $(p\neq q)$, $\limsup_{n\to\infty}|f^n(p)-f^n(q)|>0$, $\liminf_{n\to\infty}|f^n(p)-f^n(q)|=0$
2. $\forall p\in S$, periodic point $q\in I$, $\limsup_{n\to\infty}|f^n(p)-f^n(q)|>0$

Theorem (Li, Yorke [4]). *Let I be an interval and let $f: I\to I$ be continuous. Assume that f has infinitely many periodic points. Then f is chaotic.*

Definition (The order of Sharkovsky [3] [6]). Define the following ordering of the positive integers.

$$3\succ5\succ7\succ9\succ\cdots\succ2\cdot3\succ2\cdot5\succ2\cdot7\succ2\cdot9\succ\cdots\succ2^2\cdot3\succ2^2\cdot5\succ2^2\cdot7\succ$$
$$\cdots\succ2^4\succ2^3\succ2^2\succ2\succ1$$

Theorem (Sharkovsky [3] [6]). *If f has a periodic point of prime period m, then it has a periodic point of period n for every $n\prec m$.*

We consider hereafter the family of the logistic map f_λ: $[0,1]\to[0,1]$ given by

$$f_\lambda(x)=\lambda x(1-x)$$

where $1\leq\lambda\leq4$.

We remark that for any $\lambda>\lambda_\infty$, f_λ is chaotic in the sense of Li-Yorke. In fact, when $\lambda>\lambda_\infty$, f_λ has periodic points of period $2^i\cdot k$ with a positive integer i and with an odd integer k, and then it has infinitely many periodic points.

Definition (Band structure). For some value of $\lambda>\lambda_\infty$, f_λ has an m bands structure if there exist m disjoint intervals Γ_i such that

$$\frac{1}{2}\in\Gamma_1,\quad f_\lambda(\Gamma_i)=\Gamma_{i+1}\ (1\leq i\leq m-1),\quad f_\lambda(\Gamma_m)=\Gamma_1.$$

These disjoint intervals are called chaotic bands.

We define λ_{M_i} as the parameters values at which 2^i chaotic bands split into 2^{i+1} chaotic bands as the parameter λ is decreased. See Fig. 2.

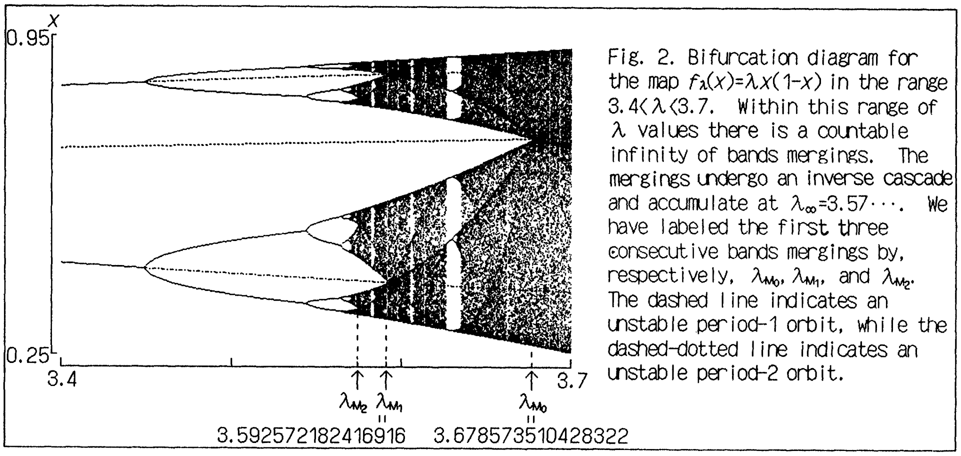

Fig. 2. Merger of chaotic bands.

We shall derive the exact parameter value at which two bands merging appears for the first time.

Theorem (Bands merging parameter). *There exists a set Λ of parameters such that for $\lambda\in\Lambda$, f_λ has no periodic point of odd period, except fixed point* (*period* 1), *and at least two bands structure. Λ is bounded and its supremum denoted by λ_{M_0} is given as the only real root of the equation,*

$$\lambda^3-2\lambda^2-4\lambda-8=0. \tag{1}$$

Proof. Assume that $\lambda>\lambda_\infty$ (>2). Let $p_\lambda=(\lambda-1)/\lambda$ $(>1/2)$, one of the fixed points for f_λ, and

$$I_1=(1-p_\lambda,p_\lambda),\quad I_n=f_\lambda(I_{n-1})\quad(n\geq1).$$

If $I_3\subset I_1$ is satisfied, then $I_{2n}\subset I_2$ and $I_{2n-1}\subset I_1$ for $n\geq1$. Since $I_1\cap I_2=\emptyset$, f_λ has no periodic point of odd period on $I_1\cup I_2$, and further, f_λ has no periodic point on $(0,1-p_\lambda]\cup(f_\lambda(1/2),1)$.

$I_3\subset I_1$ implies that f_λ has no periodic point of odd period, except period one. Moreover, if $\lambda>\lambda_\infty$, I_3 and I_2 are chaotic bands for f_λ, since it is easy to find that $1/2\in I_3$, $f_\lambda(I_3)=I_2$, $f_\lambda(I_2)=I_3$.

The parameter λ which satisfies $I_3\subset I_1$ are obtained as follows.

$$f_\lambda^2\left(\frac{1}{2}\right)>1-p_\lambda=\frac{1}{\lambda}$$

$$(\lambda-2)(\lambda^3-2\lambda^2-4\lambda-8)<0.$$

Then supremum of Λ is given as the only real root of the equation, $\lambda^3-2\lambda^2-4\lambda-8=0$;

$$\sup \Lambda=\frac{(72\sqrt{11}+152\sqrt{3})^{2/3}+2\sqrt[6]{3}(72\sqrt{11}+152\sqrt{3})^{1/3}+16\sqrt[3]{3}}{3\sqrt[6]{3}(72\sqrt{11}+152\sqrt{3})^{1/3}}$$
$$=3.678573510428322\cdots$$
$$>\lambda_\infty.$$

For $\lambda \in \Lambda$, the two intervals,

$$\Gamma_1=\left[f_\lambda^2\left(\frac{1}{2}\right),\ f_\lambda^4\left(\frac{1}{2}\right)\right],\qquad \Gamma_2=\left[f_\lambda^3\left(\frac{1}{2}\right),\ f_\lambda\left(\frac{1}{2}\right)\right]$$

are included by disjoint intervals I_3, I_2 respectively.

When $\lambda=\sup \Lambda$, critical point $1/2$ is eventually fixed; i.e.

$$p_\lambda=f_\lambda^3\left(\frac{1}{2}\right)=f_\lambda^4\left(\frac{1}{2}\right).$$

If $\lambda>\sup \Lambda$, then

$$f_\lambda^3\left(\frac{1}{2}\right)<f_\lambda^4\left(\frac{1}{2}\right),$$

and Γ_1, Γ_2 are no longer disjoint and the two chaotic bands have merged into a single band. Therefore, $\sup \Lambda$ is exactly λ_{M_0}, the parameter value at which two chaotic bands merge into one. Q.E.D.

We remark that in [5] the value λ_{M_0} is found as the solution of the same equation as (1) which is derived from the ergodic observation.

By the same consideration for $\{f_\lambda^2\}$ on the small interval $[1-p_\lambda, p_\lambda]$, there exists λ_{M_1} at which two chaotic bands for $\{f_\lambda^2\}$ merge into one; i.e. four chaotic bands for $\{f_\lambda\}$ merge into two at λ_{M_1}.

Corollary. *There exists a set Λ' of parameters such that for $\lambda \in \Lambda'$, f_λ has at least four bands structure. Λ' is bounded and its supremum is given as*

$$\lambda_{M_1}=3.5925721824169 16\cdots.$$

Proof. λ_{M_1} is obtained as the one of the roots of the following equation,

$$(f_\lambda^2)^2\left(\frac{1}{2}\right)=1-\frac{\lambda+1-\sqrt{\lambda^2-2\lambda-3}}{2\lambda}.$$

At λ_{M_1}, $(f_{\lambda_{M_1}}^2)^3(1/2)$ falls on periodic point of period two; i.e. critical point $1/2$ is eventually periodic of period two. Q.E.D.

We observe that all features are repeated in a regular fashion, by superposition of a large periodic motion of period 2^k, on the small scale. We mark λ_{M_0} at which $f_{\lambda_{M_0}}^3(1/2)$, falls on an unstable fixed point, and λ_{M_1} at which $f_{\lambda_{M_1}}^{3\cdot 2}(1/2)$ falls on an unstable point of period two. Now, there will be at the appropriately chosen value of the parameter λ_{M_2} a map such that $f_{\lambda_{M_2}}^{3\cdot 2^2}(1/2)$ falls on an unstable point of period four, etc.

References

[1] P. Collet, J.-P. Eckmann and O. E. Lanford III: Universal properties of maps on an interval. Commun. Math. Phys., **76**, 211–254 (1980).

[2] P. Collet and J.-P. Eckmann: Iterated Maps on the Interval as Dynamical Systems. Birkhäuser (1980).

[3] J. Guckenheimer: On the bifurcation of maps of the interval. Inventiones Math., **39**, 165–178 (1977).

[4] T. Y. Li and J. A. Yorke: Period three implies chaos: The American Mathematical Monthly, vol. 82, pp. 985–992 (1975).

[5] D. Ruelle: Applications conservant une mesure absolument continue par rapport à dx sur [0,1]. Commun. Math. Phys., **55**, 47–51 (1977).

[6] A. N. Sharkovsky: Co-existence of the cycles of a continuous mapping of the line into itself. Ukranian Math. J., **16**, no. 1, 61–71 (1964).

[7] M. Yoshimoto: Iteration theory of families of the 1-parameter polynomials. Master Thesis, Sophia University (1991).