38. *On the Sums of Digits in Integers*

By Hiroyuki KANO
Department of Mathematics, Faculty of Science and Technology, Keio University

(Communicated by Shokichi IYANAGA, M. J. A., May 13, 1991)

Let $r \geqq 2$ be a given integer and let

$$n = a_k a_{k-1} \cdots a_0 = a_k r^k + a_{k-1} r^{k-1} + \cdots + a_0, \qquad a_h \in \{0, 1, \cdots, r-1\}$$

be the r-adic expansion of a nonnegative integer n. We define the sum of digit function

$$S_r(n) = a_k + a_{k-1} + \cdots + a_0.$$

This function has been studied by many authors (cf. Stolarsky [2]).

Clements and Lindström [1] proved the following formula:

$$\sum_{m \leqq n} s_2(m) = \log_2 |\det A_2(n)| \qquad (n = 0, 1, 2, \cdots),$$

where

$$A_2(n) = (a_{ij})_{0 \leqq i, j \leqq n}, \qquad a_{ij} = (-1)^{\alpha_{ij}},$$

and α_{ij} is the number of common terms $b_h = c_h = 1$ in the dyadic expansions $i = \sum_{h \geqq 0} b_h 2^h$, and $j = \sum_{h \geqq 0} c_h 2^h$. In the present paper, we generalize this formula for any given base $r \geqq 2$.

We first define a matrix $A_r(n)$ for a given integer $r \geqq 2$ and any nonnegative integer n. We choose any real number ρ satisfying $1 < \rho \leqq 2^{1/(r-1)}$ and define a complex number $\zeta = \zeta(\rho)$ by

$$|\zeta| = 1, \qquad |\zeta - 1| = \rho \quad \text{with} \quad \operatorname{Im} \zeta \geqq 0.$$

(If $r = 2$, we can choose $\rho = 2$ and so $\zeta = -1$.) Then we choose real numbers $\beta_h = \beta_h(\rho)$ $(h = 1, 2, \cdots, r-1)$ such that

$$|\zeta^{\beta_h} - 1| = \rho^h \quad \text{with} \quad 1 = \beta_1 < \beta_2 < \cdots \beta_{r-1} \leqq \frac{\pi}{\rho}.$$

Let $i = \sum_{h \geqq 0} b_h r^h$ and $j = \sum_{h \geqq 0} c_h r^h$ be the r-adic expansions of nonnegative integers i and j, and put

$$A_r(n) = (a_{ij})_{0 \leqq i, j \leqq n}, \qquad a_{ij} = \zeta^{\beta_{ij}},$$

where

$$\beta_{ij} = \sum_{h \geqq 0, b_h = c_h \neq 0} \beta_{b_h}.$$

Theorem. *We have*

$$\sum_{m \leqq n} s_r(m) = \log_\rho |\det A_r(n)| \qquad (n = 0, 1, 2, \cdots).$$

Proof. For any integer $n \geqq 0$ we write

$$(1) \qquad n = hr^k + m \quad (k, h, m \in Z, k \geqq 0, 0 \leqq h < r, 0 \leqq m < r^k).$$

If we put $f(n) = \sum_{m \leqq n} s_r(m)$, then we have

$$(2) \qquad f(n) = f(hr^k + m) = hf(r^k - 1) + \frac{h(h-1)}{2} r^k + h(m+1) + f(m).$$

Conversely, this formula (2) defines the function $f(n)$ $(n \geqq 0)$ uniquely under the condition

(3) $$f(0)=0.$$

Hence it is enough to show that the function

$$f(n)=\log_\rho|\det A_r(n)| \qquad (n\geqq 0)$$

satisfies (2) and (3). The proof will be carried out by induction on n.

First if $k=0$ in (1), then $0\leqq n=h<r$ and so (2) with (3) implies

$$f(h)=\frac{h(h-1)}{2} \qquad (0\leqq h<r).$$

On the other hand, we have by the definition of $A_r(h)$ with $0\leqq h<r$.

$$A_r(h)=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta^{\beta_1} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & \zeta^{\beta_h} \end{bmatrix}.$$

Hence we obtain

$$|\det A_r(h)|=\prod_{k=1}^{h}|\zeta^{\beta_k}-1|=\rho^{h(h+1)/2} \quad (1\leqq h<r), \qquad \det A(0)=1.$$

Now let $k\geqq 1$. We assume that (2) holds for all $n=hr^{k-1}+m$ $(0\leqq h<r, 0\leqq m<r^k)$. By definition, we have, for any integers $b(0\leqq b<r)$ and i, j $(0\leqq i, j<r^k)$,

$$\beta_{br^k+i,j}=\beta_{i,j}=\beta_{i,br^k+j}, \qquad \beta_{br^k+i,br^k+j}=\beta_b+\beta_{i,j},$$

so that

$$A_r(hr^k+m)=\begin{bmatrix} B & B & \cdots & B & C \\ B & \zeta^{\beta_1}B & \cdots & B & C \\ \vdots & \vdots & & \vdots & \vdots \\ B & B & \cdots & \zeta^{\beta_{n-1}}B & C \\ D & D & \cdots & D & \zeta^{\beta_n}A_r(m) \end{bmatrix},$$

where $B=A_r(r^k-1)$, C is the $r^k\times(m+1)$-matrix consisting of the first $m+1$ columns of B, and D is the $(m+1)\times r^k$-matrix consisting of the first $m+1$ rows of B. Hence

$$\det A_r(hr^k+m)=\det\left[\begin{array}{cccc|c} B & & & & \\ B & (\zeta^{\beta_1}-1)B & & 0 & \\ \vdots & & \ddots & & 0 \\ B & 0 & & (\zeta^{\beta_{n-1}}-1)B & \\ \hline D & 0 & & & (\zeta^{\beta_n}-1)A_r(m) \end{array}\right].$$

Therefore we obtain

$$|\det A_r(hr^k+m)|=|\zeta^{\beta_1}-1|^{r^k}\cdots|\zeta^{\beta_{n-1}}-1|^{r^k}|\zeta^{\beta_n}-1|^{m+1}|\det A_r(r^k-1)|^h|\det A_r(m)|$$
$$=\rho^{((h+1)/2)r^k+h(m+1)}|\det A_r(r^k-1)|^h|\det A_r(m)|.$$

This completes the proof of the theorem.

Acknowledgement. We thank Prof. I. Shiokawa for his valuable suggestions which have improved this paper.

References

[1] G. F. Clements and B. Lindström: A sequence of (±)-determinants with large values. Proc. AMS., **16**, 548–550 (1965).
[2] K. B. Stolarsky: Power and exponential sums of digital sums related to binomial coefficient parity. SIAM J. Appl. Math., **32**, 717–730 (1977).