26. *Chiral Anomaly on a Spin Manifold*

By Hiroshi WATANABE*) and Akira YOSHIOKA**)

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1. The aim. We propose a rigorous framework to deduce the chiral anomaly on a compact spin manifold: we show that a certain functional integral describing the second quantized fermion gives this anomaly.

To this end, based on triangulations of the manifold, we define finite dimensional approximations (the lattice regularization) of the functional integral and take the continuum limit. The limit is shown to be a topological invariant by means of the index theorem.

This work is the continuation of [1], which deals only with the case of a flat torus. In the present work, we improve the formulation in [1] by substituting both the Wilson term and the regularizing heat kernel by a single operator in the action. See [2] for details.

2. Chiral anomaly. Let M be a compact oriented Riemannian spin manifold of dimension d (=even) and P a principal bundle over M with group G. Let ρ be a unitary representation of G on $\boldsymbol{C}^N$ giving the associated vector bundle $E=P\times_G \boldsymbol{C}^N$ with a Hermitian inner product $\langle\ ,\ \rangle$. We now fix a connection A of P and denote by $\boldsymbol{D}$ the Dirac operator defined by A. $\boldsymbol{D}$ acts on $\Gamma(S\otimes E)$, the space of all smooth sections of $S\otimes E$, where S is the spinor bundle over M.

Let e_μ ($\mu=1, \cdots, d$) be orthonormal vector fields around x. Let $J^\mu(x)$ be the chiral current $\langle\psi(x), \tau e_\mu \psi(x)\rangle$, where $\psi\in\Gamma(S\otimes E)$, $\tau=i^{d/2}e_1e_2\cdots e_d$ and products mean the Clifford multiplications. Then, one has a vector field J on M given by

$$J(x)=\sum_{\mu=1}^{d} J^\mu(x)e_\mu(x).$$

A solution $\psi(x)$ to the Dirac equation $\boldsymbol{D}\psi(x)=0$ is subject to the conservation law:

$$\text{div}\, J(x)=0. \tag{1}$$

The chiral anomaly is the breakdown of the conservation law (1) after the second quantization: according to physical literatures (see e.g. [3]), *the vacuum expectation* $\langle \text{div}\, J\rangle$ in general does not vanish, though the field operator $\psi(x)$ satisfies $\boldsymbol{D}\psi(x)=0$. Moreover, the vacuum expectation of

$$Y(x)=\text{div}\, J(x)-2im\langle\psi(x), \tau\psi(x)\rangle$$

can be identified with the characteristic class that appears in the index formula [4], where $m>0$ is the fermion mass. Our problems are to define and

*) Department of Mathematics, Nippon Medical School.

**) Department of Mathematics, Faculty of Science and Technology, Science University of Tokyo.

to calculate the vacuum expectation $\langle Y(x)\rangle$ with a mathematical rigor based on the lattice regularized functional integral.

3. The lattice Dirac operator. For each triangulation M of M, there exists an hermitian lattice approximation D of the Dirac operator $\boldsymbol{D}$ [2]. Every eigenfunction of $\boldsymbol{D}$ is approximated by a (linear combination of) lattice eigenfunction(s) of D with almost the same eigenvalue(s), as long as the triangulation is sufficiently fine. The opposite, however, is not the case: the difference operator D has pathological eigenfunctions that oscillate very intensely regardless of their small eigenvalues and have no continuum counterparts. Especially for a bundle with nonzero index, we see that such a pathology clearly exists, since the equality $D\tau+\tau D=0$ implies

$$\operatorname{Ind} D=\operatorname{Tr}\tau=0\neq\operatorname{Ind}\boldsymbol{D}. \tag{2}$$

This phenomenon is known to physicists under the name "species doubling" (especially in the case of flat torus).

In order to convert (2) into the equality giving $\operatorname{Ind}\boldsymbol{D}$, one has to introduce a lattice approximation W of the operator $\boldsymbol{D}^2$ so that W takes extraordinary large values for the pathological eigenfunctions (of course, D^2 is not appropriate). Then we have the following equality [2]:

$$\lim_{t\to 0}\lim_{\dot{M}\to M}\operatorname{Tr}(e^{-tW}\tau)=\operatorname{Ind}\boldsymbol{D}.$$

In the above, a suitable diagonal limit will be allowed.

4. Quantization. In the second quantized theory of fermion in the Euclidean framework, the vacuum expectation is defined as the continuum limit ($\dot{M}\to M$) of the following type of lattice regularized functional integral [5]

$$\langle\cdot\rangle_{\dot{M}}=Z_{\dot{M}}^{-1}\int\cdots\int\prod_{x\in\dot{M}}d\psi'_x\prod_{x\in\dot{M}}d\psi_x\cdot\exp\{-S_{\dot{M}}(\psi',\psi)\}, \tag{3}$$

where ψ'_x and ψ_x are Grassmann variables, $Z_{\dot{M}}$ is the normalization constant, i.e.

$$Z_{\dot{M}}=\int\cdots\int\prod_{x\in\dot{M}}d\psi'_x\prod_{x\in\dot{M}}d\psi_x\exp\{-S_{\dot{M}}(\psi',\psi)\},$$

and $S_{\dot{M}}(\psi',\psi)$ is a lattice regularized classical action.

The problem is the choice of $S_{\dot{M}}(\psi',\psi)$. In physical literatures, various prescriptions have been made: for example, Wilson (see [5]) proposed

$$S_{\dot{M}}(\psi',\psi)=\sum_{x\in\dot{M}}\langle\psi'_x,((iD+mI+aW)\psi)_x\rangle|\Delta_x|, \tag{4}$$

where I is the identity operator, $a=a_{\dot{M}}>0$ is a constant bounding the distances between all pairs of nearest neighbour points of $\dot{M}$ and $|\Delta_x|$ is the volume of star neighbourhood around x. *The Wilson term* aW in the right hand side of (4) statistically suppresses the pathological configurations in the functional integral (3).

The above choice, however, does not lead us to the complete mathematical rigor, in spite of the view expressed e.g. in [5] [6]. Other choices in physical literatures also do not improve the situation. We then propose the following action:

$$S_{t,\dot{M}}(\psi', \psi) = (\psi', (iD + mI)e^{tW}\psi), \tag{5}$$

where (,) is an appropriate inner product (see [2]) and $t>0$. This action enables us to achieve the mathematical rigor and at the same time to utilize the index theorem.

As a result, we obtain the following theorem:

Theorem. *Let $Y_{\dot{M}}(x)$ be the lattice approximation of $Y(x)$. Then it holds that*

$$\lim_{t\to 0}\lim_{\dot{M}\to M} \langle Y_{\dot{M}}(x)\rangle_{t,\dot{M}} = 2(2\pi i)^{-d/2}(\hat{a}(M)ch(E))_{d/2}(x), \tag{6}$$

where $\langle\ \rangle_{t,\dot{M}}$ is defined by the right hand side of (3) *with $S_{\dot{M}}(\psi', \psi)$ replaced by $S_{t,\dot{M}}(\psi', \psi)$, $\hat{a}(M)$ denotes the characteristic class associated with the index of Dirac operator and $ch(E)$ denotes the Chern class.*

In the left hand side of (6), a suitable diagonal limit is allowed: for example, we can put $t=a_{\dot{M}}^{\varepsilon}$ for a sufficiently small constant $\varepsilon>0$ independent of $\dot{M}$. For the precise meaning of $\lim_{\dot{M}\to M}$, see [2].

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