73. A Uniqueness Set for Linear Partial Differential Operators with Real Coefficients

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(Communicated by Shokichi IYANAGA, M. J. A., Nov. 9, 1990)

1. Introduction. Let d be a positive integer and $d\geqslant 2$. $\mathcal{O}(C^d)$ denotes the space of holomorphic functions on C^d. Suppose P is an arbitrary irreducible homogeneous polynomial with real coefficients. For any complex number λ we put $\mathcal{O}_\lambda(C^d)=\{F\in\mathcal{O}(C^d)\,;\,(P(D)-\lambda)F=0\}$. Let $\mathcal{N}=\{z\in C^d\,;\,P(z)=0\}$. The space $\mathcal{O}(\mathcal{N})$ of holomorphic functions on the analytic set $\mathcal{N}$ is equal to $\mathcal{O}(C^d)|_{\mathcal{N}}$ by the Oka-Cartan theorem.

Consider the restriction mapping $\alpha_\lambda : F\rightarrow F|_{\mathcal{N}}$ of $\mathcal{O}_\lambda(C^d)$ to $\mathcal{O}(\mathcal{N})$. In our previous paper [5] we showed that α_λ is a linear isomorphism of $\mathcal{O}_\lambda(C^d)$ onto $\mathcal{O}(\mathcal{N})$ when $P(z)=z_1^2+\cdots+z_d^2$ $(d\geqslant 3)$. In this sense we called the cone $\{z\in C^d\,;\,z_1^2+\cdots+z_d^2=0\}$ a uniqueness set for the differential operator $\sum_{j=1}^d(\partial/\partial z_j)^2+\lambda^2$ (for the case $P(z)=z_1^2+\cdots+z_d^2$, see also [4] and see [3] for more general polynomials of degree 2).

In this paper we will show that α_λ is a linear isomorphism of $\mathcal{O}_\lambda(C^d)$ onto $\mathcal{O}(\mathcal{N})$ for any $\lambda\in C$ if P is an arbitrary irreducible homogeneous polynomial with real coefficients.

2. Statement of the result and its proof. Let P be an arbitrary homogeneous polynomial and we define the polynomial P^* by $P^*(z)=\overline{P(\bar{z})}$ $(z\in C)$. $P(C^d)$ denotes the space of polynomials on C^d and $H_k(C^d)$ denotes the space of homogeneous polynomials of degree k on C^d. We define the inner product $\langle\ ,\ \rangle$ on $P(C^d)$ by the following formula:

$$\langle z^\alpha, z^\beta\rangle=\begin{cases}0 & (\alpha\neq\beta)\\ \alpha\,! & (\alpha=\beta).\end{cases}$$

We put $\mathcal{H}_k=\{F\in H_k(C^d)\,;\,P^*(D)F=0\}$ and $J_k=\{P\phi\in H_k(C^d)\,;\,\phi$ is some homogeneous polynomial on $C^d\}$. The following lemma is known.

Lemma 2.1 ([1] and [2] Theorem 3). (i) For any nonnegative integer k we have $H_k(C^d)=\mathcal{H}_k\oplus J_k$ and $\mathcal{H}_k\perp J_k$ with respect to the inner product $\langle\ ,\ \rangle$.

(ii) For any $\lambda\in C$ and any $F\in\mathcal{O}(C^d)$ there exist $H, G\in\mathcal{O}(C^d)$ uniquely such that

$$F=H+PG \tag{2.1}$$

and

$$(P^*(D)+\lambda)H=0. \tag{2.2}$$

Suppose $F\in\mathcal{O}(C^d)$. Let $F(z)=\sum_{k=0}^\infty F_k(z)$ be the development of F in a series of homogeneous polynomials F_k of degree k. Then $\sum_{k=0}^\infty F_k$ converges to F uniformly on each compact set on C^d and F_k is given by the following formula:

$$F_k(z)=\frac{1}{2\pi i}\oint_{|t|=\rho}\frac{F(tz)}{t^{k+1}}dt \qquad \text{for } z\in C^d, \tag{2.3}$$

where $\rho>0$ and the right hand side of (2.3) does not depend on ρ.

The purpose of this paper is to prove the following

Theorem 2.2. *Suppose P is an arbitrary irreducible homogeneous polynomial with real coefficients and λ is a complex number. Then the restriction mapping $F\to F|_{\mathcal{N}}$ defines the following bijection:*

$$\alpha_\lambda : \mathcal{O}_\lambda(C^d) \xrightarrow{\sim} \mathcal{O}(\mathcal{N}). \tag{2.4}$$

In order to prove the theorem we need the following

Lemma 2.3. *Let Q be an irreducible polynomial on C^d. If $h\in P(C^d)$ and h vanishes on $\{z\in C^d\,;\,Q(z)=0\}$, then there exists $g\in P(C^d)$ such that $h=Qg$.*

Lemma 2.3 can be proved by Hilbert's Nullstellensatz. We omit here the proof of this lemma.

Proof of Theorem 2.2. Let $f\in\mathcal{O}(\mathcal{N})$. Then there exists some $F\in\mathcal{O}(C^d)$ such that $F=f$ on $\mathcal{N}$ because $\mathcal{O}(\mathcal{N})=\mathcal{O}(C^d)|_{\mathcal{N}}$. We have $P=P^*$ since P has real coefficients and from Lemma 2.1 (ii) there exist $H\in\mathcal{O}_\lambda(C^d)$ and $G\in\mathcal{O}(C^d)$ uniquely such that $F=H+PG$. So $f(z)=F(z)=H(z)$ on $\mathcal{N}$ and this shows that $\alpha_\lambda H=f$. Therefore α_λ is surjective.

Next, assume that $P\in H_r(C^d)$. Suppose $F\in\mathcal{O}_\lambda(C^d)$ and $\alpha_\lambda F=0$. If we put $F=\sum_{n=0}^\infty F_n$ ($F_n\in H_n(C^d)$, $n=0,1,2,\cdots$) then there exist $H_n\in\mathcal{H}_n$ and $G_n\in H_n(C^d)$ such that

$$F_n=\begin{cases}H_n+GP_{n-r} & (n\geqslant r)\\ H_n & (0\leqslant n<r)\end{cases} \tag{2.5}$$

by Lemma 2.1 (i). Since $\sum_{n=0}^\infty F_n$ converges to F uniformly and $P\in H_r(C^d)$ we have $P(D)F=\sum_{n=0}^\infty P(D)F_n$ and $P(D)F_n\in H_{n-r}(C^d)$. Furthermore we have $P(D)F_n=\lambda F_{n-r}$ because $F\in\mathcal{O}_\lambda(C^d)$ and $P(D)F=\lambda F=\lambda\sum_{n=0}^\infty F_n$. Therefore (2.5) gives

$$P(D)PG_{n-r}=\begin{cases}\lambda H_{n-r}+\lambda PG_{n-2r} & (n\geqslant 2r)\\ \lambda H_{n-r} & (r\leqslant n<2r).\end{cases} \tag{2.6}$$

By assumption we have $F=0$ on $\mathcal{N}$. So for any nonnegative integer n we obtain $F_n=0$ on $\mathcal{N}$ by (2.3) and this shows that $H_n=0$ on $\mathcal{N}$. Hence H_n vanishes because $H_n\in J_n\cap\mathcal{H}_n=\{0\}$ from Lemma 2.3 and Lemma 2.1 (i). Therefore we have

$$P(D)PG_{n-r}=0 \qquad (r\leqslant n<2r). \tag{2.7}$$

(2.7) implies $PG_{n-r}\in\mathcal{H}_n$ and we have

$$PG_{n-r}=0 \qquad (r\leqslant n<2r). \tag{2.8}$$

From (2.8) and (2.6) we obtain $P(D)PG_{n-r}=0$ $(2r\leqslant n<3r)$ and hence $PG_k=0$ for any nonnegative integer k by iterating this. Therefore $F=0$ and α_λ is injective. Q.E.D.

Remark. In Theorem 2.2 the condition that P is irreducible is necessary. For example, consider $P(z)=(z_1^2+z_2^2+\cdots+z_d^2)^2$ and $f(z)=z_1^2+\cdots+z_d^2$. Then α_0 is not injective since $f\in\mathcal{O}_0(\boldsymbol{C}^d)$ and $f=0$ on $\mathfrak{N}$ though $f\not\equiv 0$ on $\boldsymbol{C}^d$.

References

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