# 53. *On the Local Zeta Functions of the Hilbert Modular Schemes*

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**Abstract:** We announce that the local zeta functions of suitable smooth compactifications of Hilbert modular schemes with respect to totally real algebraic number fields $K$ at prime numbers $p$ of good reduction are expressed in terms of the actions of the Hecke rings on their $l$-adic cohomology groups if $p$ remains prime in $K$. These actions of the Hecke rings have been discovered in our previous papers: Hatada [10], [11] and [12]. We announce also our estimates for the absolute values of the eigenvalues of those endomorphisms for the Hecke rings of the cohomology groups. Details will appear elsewhere.

**§1. Results.** Let $K$ be a totally real algebraic number field and let $g=[K:\boldsymbol{Q}]$. For simplicity we assume that the strict class number of $K$ is 1, i.e., any non-zero fractional ideal of $K$ is generated by a totally positive element of $K$. (For example $K=\boldsymbol{Q}(\sqrt{5})$.) We may remove this assumption. Let $\mathcal{R}$ be a commutative ring. Write

$M_{2,2}(\mathcal{R})=$the ring of the $2\times 2$ matrices with coefficients in $\mathcal{R}$;

$\mathfrak{O}_K=$the ring of the algebraic integers in $K$.

Let $N\geq 3$ be a rational integer. Write

$$GL^+(2, K)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_{2,2}(K)\,|\,ad-bc \text{ is a totally positive number.}\right\};$$

$$\Gamma(1)_K=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_{2,2}(\mathfrak{O}_K)\,|\,ad-bc \text{ is a totally positive unit of } \mathfrak{O}_K.\right\};$$

$$\Gamma(N)_K=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(1)_K\,|\,a-1\equiv d-1\equiv b\equiv c\equiv 0 \pmod{N\mathfrak{O}_K}.\right\};$$

$\mathfrak{H}^g=$the cartesian product of $g$ copies of the complex upper half plane $\mathfrak{H}$. One sees that the group $\{\varepsilon\gamma\,|\,\gamma\in \mathrm{SL}(2, \mathfrak{O}_K).\ \varepsilon \text{ is a unit of } \mathfrak{O}_K.\}$ is a finite index subgroup of $\Gamma(1)_K$. We let $\mathrm{GL}^+(2, K)$ act on $\mathfrak{H}^g$ in the usual manner (see e.g. Shimizu [18]). One has the complex analytic quotient space $\Gamma(N)_K\backslash\mathfrak{H}^g$. Let $\varphi$ be the Euler phi function. By the theory of moduli of abelian varieties with real multiplication by $\mathfrak{O}_K$ with level $N$ structure (cf. Rapoport [17]) there exists a scheme $\mathcal{M}(N)$ of moduli over $\operatorname{Spec}\boldsymbol{Z}[1/N]$ such that $\mathcal{M}(N)\times_{\operatorname{Spec}\boldsymbol{Z}[1/N]}\operatorname{Spec}\boldsymbol{C}=\bigcup_{i=1}^{\varphi(N)}\Gamma(N)_K\backslash\mathfrak{H}^g$. This right side is a disjoint union of $\varphi(N)$ copies of $\Gamma(N)_K\backslash\mathfrak{H}^g$. These copies correspond to the $N$-th roots of unity bijectively. Write $G_2^+(\mathfrak{O}_K)=\mathrm{GL}^+(2, K)\cap M_{2,2}(\mathfrak{O}_K)$ which is a monoid. Write $HR(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))=$the Hecke ring with respect to the pair $(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))$. (For the definition of Hecke rings, see e.g. Shimura [19].) By the theory of Ash *et al.* [2] and Hirzebruch [14] we choose suitable toroidal compactifications of the $\Gamma(N)_K\backslash\mathfrak{H}^g$. Namely for the compactifications of $\{\Gamma(N)_K\backslash\mathfrak{H}^g\}_{N\geq 3}$ we fix a regular and projective $\Gamma(1)_K$-admissible family of polyhedral cone decomposition: $\Sigma=\{\Sigma_\alpha\}_{F_\alpha:\text{ rational components}}$ once for all as in the case of the Siegel modular variety, cf. Hatada [10], [11], [12]. Write $M(N)_C$=the smooth toroidal compactification $(\Gamma(N)_K\backslash\mathfrak{H}^g)^\sim$ of $\Gamma(N)_K\backslash\mathfrak{H}^g$ with respect to the $\Sigma$. By the theory of Rapoport [17], from the data $\Sigma$ one obtains the "compactified" scheme $M(N)$ over $\operatorname{Spec}\boldsymbol{Z}[1/N]$ of $\mathcal{M}(N)$. Let $\zeta_N$ denote a primitive $N$-th root of unity. This scheme $M(N)$ is proper and smooth over $\operatorname{Spec}\boldsymbol{Z}[1/N]$. One has

$$M(N)\times_{\operatorname{Spec}\boldsymbol{Z}[1/N]}\operatorname{Spec}\boldsymbol{C}=\bigcup_{i=1}^{\varphi(N)}M(N)_C \qquad \text{(disjoint union)},$$

and $\quad M(N)\times_{\operatorname{Spec}\boldsymbol{Z}[1/N]}\operatorname{Spec}\boldsymbol{Z}[1/N,\zeta_N]=\bigcup_{i=1}^{\varphi(N)}B(N)\quad$ (disjoint union).

This scheme $B(N)$ over $\operatorname{Spec}\boldsymbol{Z}[1/N,\zeta_N]$ is absolutely irreducible. We have $M(N)_C=B(N)\times_{\operatorname{Spec}\boldsymbol{Z}[1/N,\zeta_N]}\operatorname{Spec}\boldsymbol{C}$. For a commutative ring $\mathcal{R}$ with $\mathcal{R}\supset\boldsymbol{Z}[1/N]$ write $M(N)\otimes\mathcal{R}=M(N)\times_{\operatorname{Spec}\boldsymbol{Z}[1/N]}\operatorname{Spec}\mathcal{R}$ from now on. Let $p$ be a rational prime with $p\nmid N$, and let $\mathfrak{p}$ be a prime ideal of $\boldsymbol{Z}[1/N,\zeta_N]$ dividing $p$. Write $D(N)=B(N)\times_{\operatorname{Spec}\boldsymbol{Z}[1/N,\zeta_N]}\operatorname{Spec}(\overline{\boldsymbol{Z}[1/N,\zeta_N]/\mathfrak{p}})$ where $\bar{\boldsymbol{F}}_p=\overline{\boldsymbol{Z}[1/N,\zeta_N]/\mathfrak{p}}$. Then $D(N)$ is a proper and smooth variety over $\operatorname{Spec}\bar{\boldsymbol{F}}_p$. One has

$$M(N)\otimes\bar{\boldsymbol{F}}_p=\bigcup_{i=1}^{\varphi(N)}D(N) \qquad \text{(disjoint union)}.$$

By the theory of the canonical models of Shimura [20], the smooth irreducible variety $D(N)$ (resp. $M(N)_C$) is defined over the prime field $\boldsymbol{F}_p$ (resp. $\boldsymbol{Q}$).

By the methods found in Hatada [10], [11] and [12], we obtain the following two theorems.

**Theorem A.** *The Hecke ring $HR(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))$ acts naturally on the singular homology group $H_n(M(N)_C,\boldsymbol{Z})$ for each integer $n\geq 0$. Namely there is a natural ring homomorphism*

$$f_n: HR(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))\longrightarrow \operatorname{End}_{\boldsymbol{Z}} H_n(M(N)_C,\boldsymbol{Z})$$

*for each integer $n\geq 0$.*

**Theorem B.** *Let $l$ and $p$ be prime numbers with $p\nmid(lN)$. The Hecke ring over $\boldsymbol{Q}_l$: $HR(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l$ acts naturally on the $l$-adic cohomology group $H^n(M(N)_C,\boldsymbol{Q}_l)$ (resp. $H^n(D(N),\boldsymbol{Q}_l)$) for each integer $n\geq 0$. Namely there is a natural anti-ring homomorphism over $\boldsymbol{Q}_l$*

$$f^{(n)}: HR(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l\longrightarrow \operatorname{End}_{\boldsymbol{Q}_l} H^n(M(N)_C,\boldsymbol{Q}_l)$$

$$(\textit{resp. } f_p^{(n)}: HR(\Gamma(N)_K, G_2^+(\mathfrak{O}_K))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l\longrightarrow \operatorname{End}_{\boldsymbol{Q}_l} H^n(D(N),\boldsymbol{Q}_l))$$

*for each integer $n\geq 0$.*

Note that $H^n(M(N)_C,\boldsymbol{Q}_l)\cong H^n(D(N),\boldsymbol{Q}_l)$ for each $n\geq 0$ if $p\nmid(lN)$ (cf. SGA 4 [1]). Identifying the two cohomology groups with each other by this isomorphism, we have $f^{(n)}=f_p^{(n)}$.

Throughout this paper we denote by $p$ a prime number such that $p\mathfrak{O}_K$ is a prime ideal in $\mathfrak{O}_K$. (Example: $p\equiv\pm 2 \pmod 5$ in the case of $K=\boldsymbol{Q}(\sqrt{5})$.) Write

$$S(p\mathfrak{O}_K)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in M_{2,2}(\mathfrak{O}_K)\,|\,(ad-bc)/p \text{ is a totally positive unit of } \mathfrak{O}_K.\right\}$$

and

$$S(p\mathfrak{O}_K)(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in S(p\mathfrak{O}_K)\,|\,a-1\equiv b\equiv c\equiv 0 \pmod{N\mathfrak{O}_K}.\right\}.$$

It is easy to see that one has finite disjoint unions

$$S(p\mathfrak{O}_K)(N)=\bigcup_{i=1}^{\mu}\Gamma(N)_K\alpha_i=\bigcup_{j=1}^{\nu}\Gamma(N)_K\beta_j\Gamma(N)_K.$$

Then put

$$T(p\mathfrak{O}_K)=\sum_{j=1}^{\nu}\Gamma(N)_K\beta_j\Gamma(N)_K\in HR(\Gamma(N)_K,\, G_2^+(\mathfrak{O}_K)).$$

We obtain:

**Proposition 1.** *There is a subscheme* $\mathcal{Z}(p\mathfrak{O}_K)$ *of* $M(N)\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}M(N)$ *over* $\mathrm{Spec}\,\boldsymbol{Z}[1/N]$, *which is the correspondence of the* $T(p\mathfrak{O}_K)$. *On each component* $M(N)_{\boldsymbol{C}}\times_{\mathrm{Spec}\,\boldsymbol{C}}M(N)_{\boldsymbol{C}}$ *of* $(M(N)\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}M(N))\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}\mathrm{Spec}\,\boldsymbol{C}$, *the scheme* $\mathcal{Z}(p\mathfrak{O}_K)\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}\mathrm{Spec}\,\boldsymbol{C}$ *coincides with the correspondence defined in the same way as in Hatada* [11, p. 63, L. 4 from the bottom and p. 65, Remark 2] *and* [12].

There is some element $\sigma_p\in \mathrm{SL}_2(\boldsymbol{Z})\subset \mathrm{SL}_2(\mathfrak{O}_K)$ such that $\sigma_p\equiv\begin{pmatrix} p^{-1} & 0 \\ 0 & p\end{pmatrix}$ $\pmod N$ if $p\nmid N$. This $\sigma_p$ induces the isomorphisms $\hat{\sigma}_p$ of $\Gamma(N)_K\backslash\mathfrak{H}^g$ and $\mathcal{M}(N)$ given by the following commutative diagram.

$$\begin{array}{ccc} \mathfrak{H}^g & \xrightarrow{\ \sigma_p\ } & \mathfrak{H}^g \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \Gamma(N)_K\backslash\mathfrak{H}^g & \xrightarrow[\ \hat{\sigma}_p\ ]{} & \Gamma(N)_K\backslash\mathfrak{H}^g \end{array}$$

Since we have chosen the $\Sigma$ as a regular and projective $\Gamma(1)_K$-admissible family of polyhedral cone decomposition for the toroidal compactification of $\Gamma(N)_K\backslash\mathfrak{H}^g$ and $\mathcal{M}(N)$, this $\hat{\sigma}_p$ extends to $\tilde{\sigma_p}: M(N)_{\boldsymbol{C}}\to M(N)_{\boldsymbol{C}}$ (resp. $\tilde{\sigma_p}: M(N)\to M(N)$) as a unique isomorphism over $\boldsymbol{C}$ (resp. $\boldsymbol{Z}[1/N]$). Further $\hat{\sigma}_p$ also gives a unique isomorphism $\tilde{\sigma_p}$ of $D(N)$ (resp. $M(N)\otimes\bar{\boldsymbol{F}}_p$). Hence $\tilde{\sigma_p}$ also induces the isomorphisms $\tilde{\sigma}_{p,n}\in \mathrm{Aut}_{\boldsymbol{Q}_l}H^n(M(N)_{\boldsymbol{C}},\boldsymbol{Q}_l)$ and $[\tilde{\sigma}_{p,n}]\in \mathrm{Aut}_{\boldsymbol{Q}_l}H^n(D(N),\boldsymbol{Q}_l)$. The $\tilde{\sigma}_{p,n}$ corresponds to the $[\tilde{\sigma}_{p,n}]$ under the isomorphism $H^n(M(N)_{\boldsymbol{C}},\boldsymbol{Q}_l)\cong H^n(D(N),\boldsymbol{Q}_l)$ by the comparison theorem if $p\nmid (lN)$. We see that the isomorphism $\tilde{\sigma_p}: M(N)_{\boldsymbol{C}}\to M(N)_{\boldsymbol{C}}$ (resp. $\tilde{\sigma_p}: D(N)\to D(N)$) is defined over the prime field $\boldsymbol{Q}$ (resp. $\boldsymbol{F}_p$).

We obtain:

**Theorem 1** (*Congruence Relation*). *Assume* $p\nmid N$. *The following equality holds true as a correspondence.*

$$\mathcal{Z}(p\mathfrak{O}_K)\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}\mathrm{Spec}\,\bar{\boldsymbol{F}}_p=\mathrm{Frob}\,(p)+p^{g-1}\langle\tilde{\sigma_p}\rangle\circ{}^t(\mathrm{Frob}\,(p))$$

*on* $(M(N)\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}M(N))\times_{\mathrm{Spec}\,\boldsymbol{Z}[1/N]}\mathrm{Spec}\,\bar{\boldsymbol{F}}_p=\bigcup_{i=1}^{\varphi(N)}D(N)\times_{\mathrm{Spec}\,\bar{\boldsymbol{F}}_p}D(N)$ (*disjoint union*). *Here we use the following notations.*

$\langle\tilde{\sigma_p}\rangle$ = *the correspondence of the morphism* $\tilde{\sigma_p}: D(N)\to D(N)$ *on each component* $D(N)$ *of* $M(N)\otimes\bar{\boldsymbol{F}}_p$;

Frob$(p)$ = *the correspondence of the Frobenius p-th power endomorphism*: $D(N) \to D(N)$ *on each component* $D(N)$ *of* $M(N) \otimes \bar{F}_p$;
${}^t$(Frob$(p)$) = *the transposed correspondence of the* Frob$(p)$.

Our second theorem is:

**Theorem 2.** *Let notations be as above. Assume* $p \nmid N$. *Let* $n$ *be an integer with* $0 \leq n \leq 2g$, *and let* $\lambda_p$ *be an eigenvalue of* $f^{(n)}(T(p\mathfrak{O}_K))$, *where* $f^{(n)}$ *is the homomorphism in Theorem B. Then we obtain*

$$|\lambda_p| \leq p^{n/2} + p^{(2g-n)/2}$$

*for any Archimedean absolute value* $|\cdot|$ *with* $|2|=2$.

Our third theorem is:

**Theorem 3.** *Let notations be as above. Assume* $p \nmid (lN)$. *Let* $n$ *be an integer with* $0 \leq n \leq 2g$. *Put*

$$P_n(M(N)_C, X) = \det(X^2 - f^{(n)}(T(p\mathfrak{O}_K))X + \sigma_{\tilde{p},n} p^g).$$

*This* $P_n(M(N)_C, X)$ *is a monic polynomial in* $Z[X]$ *and does not depend on the choice of* $l$. *Further we obtain*:

$$P_n(M(N)_C, X) = \det(X - [\text{Frob}(p)]_n) \cdot \det(X - p^g[\sigma_{\tilde{p},n}][\text{Frob}(p)]_n^{-1}).$$

*Here*

Frob$(p)$ = *the Frobenius (p-th power) endomorphism of* $D(N)$;
$[\text{Frob}(p)]_n$ = *the automorphism of* $H^n(D(N), \boldsymbol{Q}_l)$ *induced by* Frob$(p)$.

Using a theorem of Deligne (Weil Conjecture) [4], we have

**Corollary of Theorem 3.** *Let notations be as in Theorem* 3. *Assume* $n \neq g$. *Write* $P_n(M(N)_C, X) = \prod_y (X - \omega_y)$. *Then*

$$\det(X - [\text{Frob}(p)]_n) = \prod_{y'} (X - \omega_{y'})$$

*where* $y'$ *runs over all* $y$ *with* $|\omega_y| = p^{n/2}$.

Our fourth theorem is:

**Theorem 4.** *Let notations be as in Theorem* 3. *For each integer* $n$ *with* $0 \leq n \leq 2g$, *we obtain*

$$P_n(M(N)_C, X) = \det(X - [\text{Frob}(p)]_n) \cdot \det(X - [\text{Frob}(p)]_{2g-n}).$$

Put $P_n^*(M(N)_C, X) = \det(1 - f^{(n)}(T(p\mathfrak{O}_K))X + \sigma_{\tilde{p},n} p^g X^2)$ for each integer $n \geq 0$.

Recall the fundamental property of the zeta function $Z(V, X)$ of a smooth projective variety $V$ defined over a finite field $\kappa$. (See e.g. Freitag and Kiehl [8].) Put $V_0 = V^{\text{Gal}(\bar{\kappa}/\kappa)}$, which is a variety over Spec $\kappa$. One defines $Z(V, X)$ to be the formal power series $\exp(\sum_{j=1}^{\infty} \theta_j j^{-1} X^j)$, where $\theta_j$ = the number of geometric points of the $V_0$ with coordinates in the field $\kappa_j$ with $[\kappa_j : \kappa] = j$ for each integer $j \geq 0$. Recall that the smooth irreducible projective variety $D(N)$ (resp. $M(N)_C$) is defined over the prime field $\boldsymbol{F}_p$ (resp. $\boldsymbol{Q}$). By a theorem of Grothendieck the zeta function $Z(D(N), X)$ of $D(N)$ over Spec $\bar{\boldsymbol{F}}_p$ has the following equality.

$$Z(D(N), X) = \prod_{n=0}^{2g} (\det(1 - [\text{Frob}(p)]_n X))^{(-1)^{n+1}}$$

Our fifth theorem is:

**Theorem 5.** *Let notations be as above. Assume $p\nmid N$. We obtain:*

$$Z(D(N), X)=\sqrt{\prod_{n=0}^{2g} P_n^*(M(N)_C, X)^{(-1)^{n+1}}}$$

*where the constant term of the right side is* 1.

This $Z(D(N), X)$ is a rational expression of $X$.

**Corollary of Theorem 5.** *Let notations be as in Theorem 5. Assume $p\nmid N$. We obtain $Z(M(N)\otimes\bar{F}_p, X)=Z(D(N), X)^{\varphi(N)}$.*

**Remark 1.** We have $\sigma_{p,n}^{\sim}=f^{(n)}(\Gamma(N)_K\sigma_p\Gamma(N)_K)$ for each integer $n\geq 0$.

**Remark 2.** For any $u\in \mathrm{End}_{\boldsymbol{Z}} H_n(M(N)_C, \boldsymbol{Z})$, let $u\otimes_{\boldsymbol{Z}}\mathrm{id.}$ denote the element of $\mathrm{End}_{\boldsymbol{Q}} H_n(M(N)_C, \boldsymbol{Q})$ through the isomorphism $H_n(M(N)_C, \boldsymbol{Z})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\cong H_n(M(N)_C, \boldsymbol{Q})$. Assume $p\nmid N$. Then the polynomial $P_n^*(M(N)_C, X)$ has the following equality for each integer $n\geq 0$. (cf. Theorems A and B.)

$$P_n^*(M(N)_C, X)=\det(1-(f_n(T(p\mathfrak{O}_K))\otimes_{\boldsymbol{Z}}\mathrm{id.})X+(f_n(\Gamma(N)_K\sigma_p\Gamma(N)_K)\otimes_{\boldsymbol{Z}}\mathrm{id.})p^gX^2)$$

Zeta functions of modular curves have been studied closely by many mathematicians. But zeta functions of higher dimensional modular varieties have not been so sufficiently investigated yet as those of the curves. We have tried to reveal arithmetic properties of the Hilbert modular schemes employing the methods found in Hatada [10], [11] and [12]. In the present investigation we have made use of the actions of the Hecke rings on the $n$-th cohomology groups of higher dimensional smooth compactifications for any integer $n\geq 0$.

## References

[1] Artin, M., Grothendieck, A., and Verdier, J. L.: Théorie des topos et cohomologie étale des schémas (SGA 4). Lect. Notes in Math., vol. 269, 270, 305, Springer, Berlin, Heidelberg (1972–1973).

[2] Ash, A. *et al.*: Smooth Compactification of Locally Symmetric Varieties. Math. Sci. Press, Brookline (1975).

[3] Deligne, P.: Formes modulaires et réprésentations $l$-adiques. Sém. Bourbaki 1968/1969, exp. 355, Lect. Notes in Math., vol. 179, Springer, Berlin, Heidelberg pp. 139–172 (1971).

[4] ——: La Conjecture de Weil, I. Inst. Hautes Études Sci. Publ. Math., **43**, 273–307 (1974).

[5] Deligne, P. and Rapoport, M.: Les schemas de modules de courbes elliptiques. Modular Functions of One Variables II. Lect. Notes in Math., vol. 349, Springer, Berlin, Heidelberg, pp. 143–316 (1973).

[6] Eichler, M.: Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion. Arch. Math., **5**, 355–366 (1954).

[7] ——: Introduction to the Theory of Algebraic Numbers and Functions. Academic Press, New York and London (1966) (Translation of Einführung in die Theorie der algebraischen Zahlen und Funktionen, Birkhäuser Verlag, Basel (1963)).

[8] Freitag, E. and Kiehl, R.: Etale Cohomology and the Weil Conjecture. Springer, Berlin, Heidelberg (1988).

[9] Hatada, K.: Siegel cusp forms as holomorphic differential forms on certain compact varieties. Math. Ann., **262**, 503–509 (1983).

[10] ——: Homology groups, differential forms and Hecke rings on Siegel modular varieties, (1986–1987). Topics in Mathematical Analysis (A volume dedicated to the memory of A. L. Cauchy) (ed. Th. M. Rassias), World Scientific Publishing Co., Singapore, pp. 371–409 (1989).

[11] ——: Correspondences for Hecke rings and $l$-adic cohomology groups on smooth compactifications of Siegel modular varieties. Proc. Japan Acad., **65**A, 62–65 (1989).

[12] ——: Correspondences for Hecke rings and (co-)homology groups on smooth compactifications of Siegel modular varieties. Tokyo J. Math., **13**, 37–62 (1990).

[13] ——: On the action of Hecke rings on homology groups of smooth compactifications of Siegel modular varieties and Siegel cusp forms. ibid., **13**, 191–205 (1990).

[14] Hirzebruch, F.: Hilbert modular surfaces. L'Ens. Math., **71**, 183–281 (1973).

[15] Namikawa, Y.: A new compactification of the Siegel space and degeneration of abelian varieties, I. Math. Ann., **221**, 57–141 (1976); II. ibid., 201–241 (1976).

[16] ——: Toroidal Compactification of Siegel Spaces. Lect. Notes in Math., vol. 812, Springer, Berlin, Heidelberg (1980).

[17] Rapoport, M.: Compactifications de l'espace de modules de Hilbert-Blumenthal. Comp. Math., **36**, 255–335 (1978).

[18] Shimizu, H.: On discontinuous groups operating on the product of upper half planes. Ann. Math., **77**, 33–71 (1963).

[19] Shimura, G.: Introduction to the Arithmetic Theory of Automorphic Functions. Iwanami Shoten, Princeton University Press; Tokyo, Princeton (1971).

[20] ——: On some arithmetic properties of modular forms of one and several variables. Ann. Math., **97**, 491–515 (1976).