52. A Note on *p*-valently Bazilević Functions

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1. Introduction. Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathcal{N}=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. A function $f(z)$ in $\mathcal{A}(p)$ is said to be p-valently starlike in $\mathcal{U}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z \in \mathcal{U}). \tag{1.2}$$

We denote by $\mathcal{S}^*(p)$ the subclass of $\mathcal{A}(p)$ consisting of all p-valently starlike functions in $\mathcal{U}$.

A function belonging to $\mathcal{A}(p)$ is said to be a member of the class $\mathcal{B}(p, \alpha)$ if there exists a function $g(z) \in \mathcal{S}^*(p)$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}\right\}>0 \tag{1.3}$$

for some α ($\alpha>0$) and for all $z \in \mathcal{U}$. Then, we note that $\mathcal{B}(p, \alpha)$ is the subclass of p-valently Bazilević functions in the unit disk $\mathcal{U}$. In particular, the class $\mathcal{B}(1, \alpha)$ when $p=1$ was studied by Singh [3], and by Obradović ([1], [2]).

2. Main result. In order to derive our result, we need the following lemma due to Obradović [2].

Lemma. *If $f(z) \in \mathcal{B}(1, \alpha)$, $\alpha>0$, then the function $F(z)$ defined by*

$$F(z)^{\alpha}=\frac{\alpha+1}{z}\int_0^z f(t)^{\alpha}dt \qquad (z \in \mathcal{U}) \tag{2.1}$$

is also in the class $\mathcal{B}(1, \alpha)$.

An application of the above lemma leads to

Main result. *If $f(z) \in \mathcal{B}(p, \alpha)$, $\alpha>0$, then the function $H(z)$ defined by*

$$H(z)^{\alpha}=\frac{p\alpha+1}{z}\int_0^z f(t)^{\alpha}dt \qquad (z \in \mathcal{U}) \tag{2.2}$$

is also in the class $\mathcal{B}(p, \alpha)$.

Proof. We note that $f(z) \in \mathcal{B}(p, \alpha)$ implies that there exists a function $g(z) \in \mathcal{S}^*(p)$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}\right\}>0 \qquad (z \in \mathcal{U}).$$

Letting $f(z)=f_1(z)^p$, $g(z)=g_1(z)^p$, and $H(z)=H_1(z)^p$, we have

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(2.3)
$$\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}=p\frac{zf_1'(z)f_1(z)^{p\alpha-1}}{g_1(z)^{p\alpha}}$$

and

(2.4)
$$H_1(z)^{\alpha}=\frac{\alpha+1}{z}\int_0^z f_1(t)^{\alpha}dt.$$

Therefore, it follows from (2.3), (2.4), and Lemma that

$$\begin{aligned} f(z)\in\mathcal{B}(p,\alpha) &\Longleftrightarrow f_1(z)\in\mathcal{B}(1,p\alpha)\\ &\Longrightarrow H_1(z)\in\mathcal{B}(1,p\alpha)\\ &\Longleftrightarrow H(z)\in\mathcal{B}(p,\alpha).\end{aligned}$$

This completes the assertion of our main theorem.

Taking $\alpha=1/p$ in our theorem, we have

Corollary. *If $f(z)\in\mathcal{B}(p,1/p)$, then the function $H(z)$ defined by*

(2.5)
$$H(z)^{1/p}=\frac{2}{z}\int_0^z f(t)^{1/p}dt \qquad (z\in\mathcal{U})$$

is also in the class $\mathcal{B}(p,(1/p))$.

References

[1] M. Obradović: Estimates of the real part of $f(z)/z$ for some classes of univalent functions. Mat. Vesnik, **36**, 266–270 (1984).
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[3] R. Singh: On Bazilević functions. Proc. Amer. Math. Soc., **38**, 261–271 (1973).