51. *A Remark on a Class of Certain Analytic Functions*

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Let A denote the class of functions of the form

(1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the unit disk $U=\{z\,;\,|z|<1\}$.

A function $f(z)\in A$ is said to be a member of the class $S(\alpha)$ if it satisfies

(2) $$\frac{zf'(z)}{f(z)}\prec 1+(1-\alpha)z$$

for some α $(0\leqq\alpha<1)$ and for all $z\in U$. The symbol $\prec$ denotes the subordination. It follows from (2) that if $f(z)\in S(\alpha)$ then $zf'(z)/f(z)$ maps the unit disk U onto the domain which is inside the open disk centered at one with radius $1-\alpha$. From this fact, we see that $f(z)\in S(\alpha)$ if and only if

(3) $$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha \qquad (z\in U).$$

We easily see that the class $S(\alpha)$ is a subclass of $S^*(\alpha)$ known as starlike of order α.

In order to derive our main result, we have to recall here the following lemma due to Jack [1] (or Miller and Mocanu [2]).

Lemma. *Let $w(z)$ be regular in the unit disk U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the cercle $|z|=r$ at point z_0, then*

$$z_0w'(z_0)=kw(z_0),$$

where k is real and $k\geqq1$.

Applying the above lemma, we have

Main theorem. If $f(z)\in A$ satisfies

(4) $$\left|\beta\left(\frac{zf'(z)}{f(z)}-1\right)+(1-\beta)\frac{z^2f''(z)}{f(z)}\right|<1-\alpha \qquad (z\in U)$$

for some α $(0\leqq\alpha<1)$, β $(0\leqq\beta<1)$, then $f(z)\in S(\alpha)$.

Proof. Defining the function $w(z)$ by

(5) $$w(z)=\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-1\right)$$

for $f(z)\in A$, we see that $w(z)$ is regular in the unit disk U and $w(0)=0$. Taking the logarithmic differentiations of both sides in (5), we have

(6) $$\frac{zf''(z)}{f'(z)}=(1-\alpha)w(z)+\frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}.$$

It follows that

(7) $$\left|\beta\left(\frac{zf'(z)}{f(z)}-1\right)+(1-\beta)\frac{z^2f''(z)}{f(z)}\right|$$

$$=\left|(1-\alpha)w(z)\left\{1+(1-\beta)(1-\alpha)w(z)+(1-\beta)\frac{zw'(z)}{w(z)}\right\}\right|$$
$$<1-\alpha.$$

Assume that there exist a point z_0 such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1.$$

Then letting $w(z_0)=e^{i\theta}$ and using lemma, we obtain

$$(8)\qquad \left|(1-\alpha)w(z_0)\left\{1+(1-\beta)(1-\alpha)w(z_0)+(1-\beta)\frac{z_0w'(z_0)}{w(z_0)}\right\}\right|$$
$$=(1-\alpha)|1+(1-\beta)(k+(1-\alpha)e^{i\theta})|$$
$$\geqq 1-\alpha$$

which contradicts our condition (4). This implies that

$$|w(z)|=\left|\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-1\right)\right|<1 \qquad (z\in U),$$

that is,

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha \qquad (z\in U).$$

Therefore, we complete the proof of our main theorem.

Taking $\beta=0$, we have

Corollary 1. *If $f(z)\in A$ satisfies*

$$(9)\qquad \left|\frac{z^2f''(z)}{f(z)}\right|<1-\alpha \qquad (z\in U)$$

for some α $(0\leqq\alpha<1)$, then $f(z)\in S(\alpha)$.

Further making $\beta=\frac{1}{2}$, we have

Corollary 2. *If $f(z)\in A$ satisfies*

$$(10)\qquad \left|\frac{zf'(z)}{f(z)}-1+\frac{z^2f''(z)}{f(z)}\right|<2(1-\alpha) \qquad (z\in U)$$

for some α $(0\leqq\alpha<1)$, then $f(z)\in S(\alpha)$.

References

[1] I. S. Jack: Functions starlike and convex of order α. J. London Math. Soc., **3**, 469–474 (1971).

[2] S. S. Miller and P. T. Mocanu: Second order differential inequalities in complex plane. J. Math. Anal. Appl., **65**, 289–305 (1978).