39. Askey-Wilson Polynomials and the Quantum Group $SU_q(2)$

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The *Askey-Wilson polynomials* are a 4-parameter family of q-orthogonal polynomials expressed by the basic hypergeometric series ${}_4\varphi_3$. As special cases, it provides various types of q-Jacobi polynomials such as little, big and continuous q-Jacobi polynomials. In this note, we report that a (partially discrete) 4-parameter family of Askey-Wilson polynomials is realized as "doubly associated spherical functions" on the quantum group $SU_q(2)$.

In [2], Koornwinder realized a 2-parameter subfamily of Askey-Wilson polynomials as *zonal* spherical functions on $SU_q(2)$ in *an infinitesimal sense.* Generalizing his arguments to *non-zonal cases,* we obtain a 4-parameter family of Askey-Wilson polynomials that are connected to these polynomials as Jacobi polynomials are to Legendre polynomials in the $SU(2)$ case. From this interpretation, we also derive an addition formula for Koornwinder's 2-parameter extension of the continuous q-Legendre polynomials. Details will be given elsewhere.

1. Throughout this note, we fix a real number q with $0<q<1$. The algebra of functions $A(G)$ on the quantum group $G=SU_q(2)$ is the C-algebra generated by x, u, v, y with fundamental relations

$$(1.1)\qquad \begin{cases} qxu=ux, \ qxv=vx, \ quy=yu, \ qvy=yv, \\ uv=vu, \ xy-q^{-1}uv=yx-qvu=1, \end{cases}$$

and the *-structure determined by $x^*=y$ and $v^*=-qu$. The quantized universal enveloping algebra $U_q(su(2))$ is the C-algebra generated by k, k^{-1}, e, f with relations

$$(1.2)\qquad \begin{cases} kk^{-1}=k^{-1}k=1, \ kek^{-1}=qe, \ kfk^{-1}=q^{-1}f, \\ ef-fe=(k^2-k^{-2})/(q-q^{-1}), \end{cases}$$

and the *-structure with $k^*=k$ and $e^*=f$. As for the Hopf algebra structure, we take the coproduct determined by

$$\Delta(k)=k\otimes k, \quad \Delta(e)=k^{-1}\otimes e+e\otimes k, \quad \Delta(f)=k^{-1}\otimes f+f\otimes k.$$

The algebra of functions $A(G)$ has a natural structure of two-sided $U_q(su(2))$-module. For each $j\in(1/2)N$, there exists a unique $2j+1$ dimensional irreducible representation of G of highest weight q^j with respect to $k\in U_q(su(2))$. By V_j we denote the corresponding right $A(G)$-comodule with coaction R: $V_j\to V_j\otimes A(G)$. We fix a C-basis $(v_m^j)_{m\in I_j}$ for V_j, with $I_j=\{j, j-1, \cdots, -j\}$, such that the differential representation takes the form

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(1.3)
$$\begin{cases} k.\, v_m^j = v_m^j q^m, \\ e.\, v_m^j = v_{m+1}^j([j-m][j+1+m])^{1/2}, \\ f.\, v_m^j = v_{m-1}^j([j+m][j+1-m])^{1/2}, \end{cases}$$

where $[m]=(q^m-q^{-m})/(q-q^{-1})$. This representation is unitary with respect to the Hermitian form $\langle\ ,\ \rangle$ on V_j such that $\langle v_m^j, v_n^j\rangle=\delta_{mn}$ ($m, n\in I_j$) and the *-operation of $U_q(su(2))$. See also [3].

2. For each matrix

(2.1)
$$g=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in GL(2;\boldsymbol{C}),$$

we define the twisted primitive element $\theta(g)\in U_q(su(2))$ by

(2.2)
$$\theta(g)=-\alpha\beta q^{-1/2}e+(\alpha\delta+\beta\gamma)(k-k^{-1})/(q-q^{-1})+\gamma\delta q^{1/2}f.$$

When $q\to 1$, the element $\theta(g)$ corresponds to a generator of the Lie algebra of the subgroup $K(g):=gKg^{-1}$ of $SU(2)$, where K is the diagonal subgroup of $SU(2)$.

Theorem 1. *Let g be a matrix of the form* (2.1) *and assume that*

$$\alpha\delta-q^{2k}\beta\gamma\neq 0 \qquad \textit{for all } k\in\boldsymbol{Z}.$$

For each $m\in(1/2)\boldsymbol{Z}$, set

(2.3)
$$\lambda_m(g)=(q^m\alpha\delta-q^{-m}\beta\gamma)(q^m-q^{-m})/(q-q^{-1}).$$

Then the element $k\theta(g)$ is diagonalizable on each left $U_q(su(2))$-module V_j ($j\in(1/2)\boldsymbol{N}$). Its eigenvalues are given by $\lambda_m(g)$ ($m=j, j-1, \cdots, -j$).

We remark that Theorem 1 is also valid when q is a nonzero complex number as long as q is not a root of unity. It is essentially the same as Theorem 8.5 of Koornwinder [2].

Hereafter, we assume that the parameter of (2.1) satisfies the condition $\bar\alpha=\delta$, $\bar\gamma=-\beta$ so that $(k\theta(g))^*=k\theta(g)$. Then we see that there exists a family of orthogonal bases $(v_m^j(g))_{m\in I_j}$ for V_j, depending polynomially on $(\alpha,\beta,\gamma,\delta)$, such that

(2.4)
$$k\theta(g).\, v_m^j(g)=v_m^j(g)\lambda_m(g) \qquad \text{for all } m\in I_j,$$

and

(2.5)
$$\langle v_m^j(g), v_n^j(g)\rangle=\delta_{mn}D_m^j(g) \qquad \text{for } m, n\in I_j,$$

where

$$D_m^j(g)=\prod_{-j-m\le k\le j-m,\, k\neq -2m}(\alpha\delta-q^{2k}\beta\gamma).$$

We fix such a family of orthogonal bases $(v_m^j(g))_{m\in I_j}$ for V_j under a suitable normalization, although we do not give here its precise description. The connection coefficients between the bases $(v_m^j)_{m\in I_j}$ and $(v_m^j(g))_{m\in I_j}$ can be written explicitly by Stanton's q-Krawtchouk polynomials (see also [2]).

3. We now introduce the matrix elements of V_j relative to the two bases $(v_m^j(g_1))_m$ and $(v_m^j(g_2))_m$. Let (g_1, g_2) be a couple of elements in $GL(2;\boldsymbol{C})$ such that

(3.1)
$$g_i=\begin{bmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{bmatrix}\in GL(2;\boldsymbol{C}); \qquad \bar\alpha_i=\delta_i,\ \bar\beta_i=-\gamma_i\ (i=1,2).$$

We define the matrix element $\varphi_{mn}^j(g_1, g_2)\in A(G)$ ($m, n\in I_j$) of V_j by

(3.2)
$$\varphi_{mn}^j(g_1, g_2):=\langle v_m^j(g_1), R(v_n^j(g_2))\rangle.$$

We also set $\psi^j_{mn}(g_1, g_2) := \varphi^j_{mn}(g_1, g_2)\cdot k$ by using the right action of $k \in U_q(su(2))$.

Proposition 2. a) *The element* $\psi = \psi^j_{mn}(g_1, g_2)$ *has the relative invariance*

$$k\theta(g_2).\psi = \psi\lambda_n(g_2) \quad \text{and} \quad \psi.\theta(g_1)k = \lambda_m(g_1)\psi. \tag{3.3}$$

b) *The elements* $\psi^j_{mn}(g_1, g_2)$ $(j \in (1/2)N,\ m, n \in I_j)$ *form an orthogonal basis for* $A(G)$ *under the Hermitian form* $\langle\ ,\ \rangle_L$ *defined by the Haar measure. The square length of* $\psi^j_{mn}(g_1, g_2)$ *is given by*

$$\langle \psi^j_{mn}(g_1, g_2), \psi^j_{mn}(g_1, g_2)\rangle_L = q^{2j}\frac{1-q^2}{1-q^{2(2j+1)}}D^j_m(g_1)D^j_n(g_2). \tag{3.4}$$

c) *For any* g, *one has*

$$\Delta(\varphi^j_{mn}(g_1, g_2)) = \sum_k D^j_k(g)^{-1}\varphi^j_{mk}(g_1, g)\otimes\varphi^j_{kn}(g, g_2). \tag{3.5}$$

In view of the relative invariance (3.3), we say that the elements $\psi^j_{mn}(g_1, g_2)$ are *doubly associated spherical functions* on G.

4. For each $m, n \in \frac{1}{2}Z$, we set

$$e_{mn}(g_1, g_2) := \psi^j_{mn}(g_1, g_2) \qquad \text{with } j = \max\{|m|, |n|\}.$$

This element is a basic relative invariant in the sense that it appears with smallest j among all relative invariants ψ satisfying (3.3). These $e_{mn}(g_1, g_2)$ are expressed as products of linear combinations of the generators x, u, v, y for $A(G)$.

The general matrix elements $\psi^j_{mn}(g_1, g_2)$ are expressed by the Askey-Wilson polynomials [1]:

$$p_n(x\,; a, b, c, d\,|\,q) = a^{-n}(ab, ac, ad\,; q)_n\, {}_4\varphi_3\left(\begin{matrix} q^{-n}, & abcdq^{n-1}, & az, & az^{-1} \\ ab, & ac, & ad \end{matrix}; q, q\right),$$

where $x = (z+z^{-1})/2$. To describe the matrix elements, we introduce the following 2-parameter extension of the continuous q-Jacobi polynomials:

$$p^{(\alpha,\beta)}_n(x\,; s, t : q) := p_n\left(x\,; \frac{t}{s}q^{1/2}, \frac{s}{t}q^{\alpha+1/2}, -\frac{1}{st}q^{1/2}, -stq^{\beta+1/2}\,|\,q\right), \tag{4.1}$$

where s and t are continuous parameters. If $(\alpha, \beta) = (0, 0)$, then formula (4.1) gives Koornwinder's 2-parameter extension of the continuous q-Legendre polynomials in [2]. If $(s, t) = (1, 1)$, (4.1) is Rahman's parametrization of continuous q-Jacobi polynomials.

For a couple (g_1, g_2) of (3.1), we define the zonal element $X = X(g_1, g_2)$ by

$$2|\alpha_1\gamma_1\alpha_2\gamma_2|X = \frac{1}{q+q^{-1}}(\psi^1_{00}(g_1, g_2) - (\alpha_1\delta_1 + \beta_1\gamma_1)(\alpha_2\delta_2 + \beta_2\gamma_2)), \tag{4.2}$$

assuming that $\alpha_i \neq 0$, $\gamma_i \neq 0$ $(i = 1, 2)$. Note that $X = X(g_1, g_2)$ satisfies

$$k\theta(g_2).X = 0, \quad X.\theta(g_1)k = 0, \quad X^* = X.$$

Theorem 3. *The doubly associated spherical functions* $\psi^j_{mn}(g_1, g_2)$ *are represented by the Askey-Wilson polynomials* (4.1) *in* X.

Case I. $m+n \geq 0$, $m \leq n$:

$$q^{-k(k+\mu+2\nu)}C_{\mu\nu k}|\alpha_1\gamma_1\alpha_2\gamma_2|^k p^{(\mu,\nu)}_k(X\,; |\alpha_2/\gamma_2|, |\alpha_1/\gamma_1| : q^2)e_{mn}(g_1, g_2),$$

Case II. $m+n \geq 0$, $m \geq n$:

$$q^{-k(k+\mu+2\nu)}C_{\mu\nu k}|\alpha_1\gamma_1\alpha_2\gamma_2|^k p^{(\mu,\nu)}_k(X\,; |\alpha_1/\gamma_1|, |\alpha_2/\gamma_2| : q^2)e_{mn}(g_1, g_2),$$

Case III. $m+n\leq 0,\ m\geq n$:

$$q^{-k(k+\mu)}C_{\mu\nu k}|\alpha_1\gamma_1\alpha_2\gamma_2|^k p_k^{(\mu,\nu)}(X\,;|\gamma_2/\alpha_2|,|\gamma_1/\alpha_1|:\ q^2)e_{mn}(g_1,g_2),$$

Case IV. $m+n\leq 0,\ m\leq n$:

$$q^{-k(k+\mu)}C_{\mu\nu k}|\alpha_1\gamma_1\alpha_2\gamma_2|^k p_k^{(\mu,\nu)}(X\,;|\gamma_1/\alpha_1|,|\gamma_2/\alpha_2|:\ q^2)e_{mn}(g_1,g_2).$$

Here $\mu=|m-n|$, $\nu=|m+n|$, $k=min\{j+m, j-m, j+n, j-n\}$ *and* $C_{\mu\nu k}$ *stands for*

$$C_{\mu\nu k}=\left(\frac{q^{2(\mu+\nu+1)}\,;\,q^2)_k}{(q^2,q^{2(\mu+1)},q^{2(\nu+1)}\,;\,q^2)_k}\right)^{1/2}.$$

Theorem 3 is a generalization of Theorem 8.3 of Koornwinder [2] to non-zonal cases. The expressions in Theorem 3 make sense even when some of the α_1, γ_1, α_2, γ_2 are zero. We also remark that the orthogonality in Proposition 2 is interpreted as the orthogonality relation for the Askey-Wilson polynomials.

By the above interpretation, we obtain an addition formula for $p_n^{(0,0)}(x\,;s,t:q)$. In fact, property (3.5) is translated into an addition formula for them.

Theorem 4. *The polynomials* $p_n^{(0,0)}(x\,;s,t:q)$ $(n\in N)$ *have the following addition formula involving an extra parameter* u:

(4.3) $$q^{-n/2}(q\,;q)_n p_n^{(0,0)}(x(zw)\,;s,t:q)$$

$$=\frac{1}{(-u^2q,-u^{-2}q\,;q)_n}p_n^{(0,0)}(x(z)\,;u,s:q)p_n^{(0,0)}(x(w)\,;u,t:q)$$

$$+\sum_{k=1}^{n}\frac{(q\,;q)_{n+k}(1+u^2q^{2k})z^{-k}w^{-k}\left(\frac{u}{s}z,-usz,\frac{u}{t}w,-utw\,;q\right)_k}{(q\,;q)_{n-k}(1+u^2)(-u^2q\,;q)_{n+k}(-u^{-2}q\,;q)_{n-k}}$$

$$\times p_{n-k}^{(k,k)}(x(z)\,;u,s:q)p_{n-k}^{(k,k)}(x(w)\,;u,t:q)$$

$$+\sum_{k=1}^{n}\frac{(q\,;q)_{n+k}(1+u^{-2}q^{2k})z^{-k}w^{-k}\left(\frac{s}{u}z,-\frac{1}{us}z,\frac{t}{u}w,-\frac{1}{ut}w\,;q\right)_k}{(q\,;q)_{n-k}(1+u^{-2})(-u^2q\,;q)_{n-k}(-u^{-2}q\,;q)_{n+k}}$$

$$\times p_{n-k}^{(k,k)}\left(x(z)\,;\frac{1}{u},\frac{1}{s}:q\right)p_{n-k}^{(k,k)}\left(x(w)\,;\frac{1}{u},\frac{1}{t}:q\right),$$

where z *and* w *are independent variables and* $x(z)=(q^{-1/2}z+q^{1/2}z^{-1})/2$.

We remark that Rahman and Verma [4] have obtained an addition formula for Rogers' q-ultraspherical polynomials $p_n^{(\alpha,\alpha)}(x\,;1,1:q)$ by analytic methods. Their work suggests that Theorem 4 may be extended to an addition formula containing one more parameter.

References

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