# 37. On Certain Homotopy-homomorphic Elements of $\pi_{n+1}(X)$

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§0. **Introduction.** Let $X$ be a topological space with a base point $x_0$ and let $\Omega(X)$ be the loop space of $X$ at $x_0$. We give $\Omega(X)$ the constant loop at $x_0$ as a base point. As well-known there exists the isomorphism: $\pi_{n+1}(X) \to \pi_n(\Omega X)$. We identify elements of these groups by this isomorphism. Now let $a$, $b$ be given integers and $\mu$: $S^n \times S^n \to S^n$ be a map of type $(a, b)$, i.e. such that $\mu(x, *)$ and $\mu(*, y)$ are maps $S^n \to S^n$ of degree $a$ and $b$ respectively. We call an element $\alpha$ of $\pi_{n+1}(X)$ a $\mu$-homomorphic element (or to be $\mu$-homomorphic) if and only if

$$\alpha(\mu(x, y)) = \omega(\alpha(m_a(x)), \alpha(m_b(y)))$$

where $\omega$ denotes the usual multiplication in $\Omega(X)$ and $m_a$ is a map: $S^n \to S^n$ of degree $a$ (in fact $m_a(x) = \mu(x, *)$).

In this note our purpose is to find an obstruction for determining to be $\mu$-homomorphic. As a result we prove

**Theorem 1.** *For an element $\alpha$ of $\pi_{n+1}(X)$, $\alpha$ is $\mu$-homomorphic if and only if $\alpha_*(c(\mu)) = 0$ where $c(\mu)$ denotes the Hopf construction as defined by James* ([2]).

An analogous problem has been considered in case of $\pi_3(G)$ for compact connected Lie groups $G$ and $(a, b) = (1, 1)$ by Takahashi ([3]).

Our obstruction defines a correspondence

$$\chi: \pi_n(\Omega(X)) \longrightarrow \pi_{2n}(\Omega(X)).$$

This correspondence $\chi$ is not neccesarily homomorphic. We prove

**Theorem 2.** *$\chi$ is homomorphic if $\Omega(X)$ is a homotopy commutative Hopf space under the usual multiplication.*

§1. **An obstruction.** Denote with $\bar{\alpha}$ the adjoint element of $\alpha \in \pi_n(\Omega(X))$, and consider two maps: $S^n \times S^n \to \Omega(X) \times \Omega(X)$ in the following diagram:

(1)
$$\begin{array}{ccccc} S^n \times S^n & \xrightarrow{\mu_1 \times \mu_2} & S^n \times S^n & \xrightarrow{\alpha \times \alpha} & \Omega(X) \times \Omega(X) \\ \mu \downarrow & & & & \downarrow \omega \\ S^n & & \longrightarrow & & \Omega(X) \end{array}$$

where $\mu_1(x) = \mu(x, *)$ and $\mu_2(y) = \mu(*, y)$. These two maps, $\alpha(\mu(x, y))$ and $\omega(\alpha(\mu_1(x)), \alpha(\mu_2(y)))$ coincides with each other on the sub-space $S^n \vee S^n$, so we have the difference element $\chi(\alpha) \in \pi_{2n}(\Omega(X))$ defined by these maps. Since $\Omega(X)$ is a Hopf space we have, from Puppe exact sequence,

**Lemma 1.** *$\alpha$ is $\mu$-homomorphic if and only if $\chi(\alpha) = 0$.*

Thus it is sufficient for our purpose to describe $\chi(\alpha)$ as stated in

Theorem 1. Let $\iota: S^n \to \Omega(S^{n+1})$ be the inclusion. First we note a decomposition of $\alpha: S^n \to \Omega(S^{n+1}) \to \Omega(X)$,

$$\alpha = (\Omega\bar{\alpha})\iota.$$

Then from the diagram (1) we obtain the diagram:

$$\text{(2)}\quad \begin{array}{ccccccc} S^n \times S^n & \longrightarrow & S^n \times S^n & \longrightarrow & \Omega(S^{n+1}) \times \Omega(S^{n+1}) & \longrightarrow & \Omega(X) \times \Omega(X) \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ S^n & & \longrightarrow & & \Omega(S^{n+1}) & \longrightarrow & \Omega(X) \end{array}$$

and we have

$$\text{(3)}\qquad \chi(\alpha) = (\Omega(\bar{\alpha})) \cdot (\chi(\iota))$$

from the naturality of difference elements.

Now we replace the space $\Omega(S^{n+1})$ by the reduced product $S^n(\infty)$ ([1]). Then we obtain the diagram:

$$\text{(4)}\quad \begin{array}{ccccc} S^n \times S^n & \longrightarrow S^n \times S^n \longrightarrow & S^n(\infty) \times S^n(\infty) \\ \downarrow{\scriptstyle \mu} & i & \downarrow \\ S^n & \longrightarrow & S^n(\infty) \end{array} \quad \Rightarrow \quad \begin{array}{ccc} S^n \times S^n & \longrightarrow & S^n \times S^n \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle q} \\ S^n & \longrightarrow & S^n(2) \end{array}$$

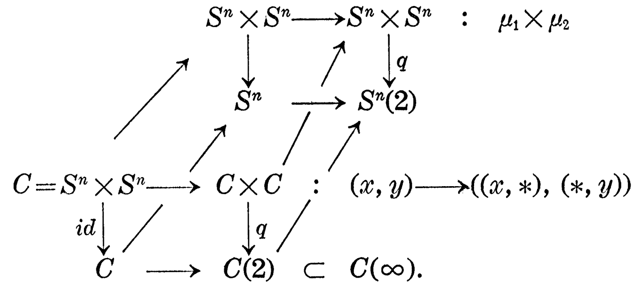
where $i$ denotes the inclusion map: $S^n \to S^n(2) \to S^n(\infty)$ and $q$ is the identification $(x, *) \equiv (*, x)$.

**Lemma 2.** *In the diagram* (4) *two maps are given by*

$$(x, y) \longrightarrow [\mu(x, y)] \quad \textit{and} \quad [\mu(x, *), \mu(*, x)].$$

*Then the difference element of these maps, i.e* $\chi(i)$ *is obtained from the Hopf construction of* $\mu$.

*Proof.* Consider a map between diagrams:

(5) 


In the lower diagram two maps are given by

$$(x, y) \longrightarrow [(x, y)] \quad \text{and} \quad [(x, *), (*, y)].$$

We denote the difference element of these maps with $d(n, n)$, then the characterization of the Hopf construction by James ([2]) shows that $d(n, n)$ is the universal example of any map from $S^n \times S^n$ to a space and therefore $\mu(\infty) \cdot (d(n, n))$ is the Hopf construction of $\mu$, so the proof is completed.

Now the proof of Theorem 1 easily follows from Lemma 2 and (5).

**§ 2. The correspondence $\chi$.** Now our obstruction $\chi$ defines a correspondence

$$\pi_{n+1}(X) \longrightarrow \pi_{2n+1}(X), \qquad \{\pi_n(\Omega(X)) \longrightarrow \pi_{2n}(\Omega(X))\}.$$

First we prove

**Lemma 3.** *For a map* $\mu: S^n \times S^n \to S^n$ *of type* $(a, b)$ *we have*

$$\chi(\alpha+\beta)=\chi(\alpha)+\chi(\beta)+ab[\alpha, \beta].$$

*where* [ , ] *denotes Whitehead product.*

*Proof.* By Theorem 1 we have

$$\chi(\alpha+\beta)=(\alpha+\beta)_*(c(\mu)).$$

Hence, a well-known formula ([4]) gives

$$\chi(\alpha+\beta)=\chi(\alpha)+\chi(\beta)+H(\mu)[\alpha, \beta]$$

where $H(\mu)$ is the Hopf invariant of $c(\mu)$. Then the proof is completed from $H(\mu)=ab$. Therefore we have

**Proposition 1.** *If all Whitehead products vanish in* $\pi_*(X)$ *then* $\chi$ *is a homomorphism.*

Now the proof of Theorem 2 follows from Lemma 3 because the formula of Lemma 3 has the adjoint form in $\pi_*(\Omega(X))$

$$\chi(\alpha+\beta)=\chi(\alpha)+\chi(\beta)+ab\langle\alpha, \beta\rangle$$

where $\langle\ ,\ \rangle$ denotes Samelson product.

Let $\mu_1$, $\mu_2$ be two maps of the same type and $\chi_i$ be the obstruction for $\mu_i$ $(i=1,2)$. We prove

**Proposition 2.** *Our obstruction is determined by the type of* $\mu$ *only, namely* $\chi_1=\chi_2$.

*Proof.* First we note that there exists a map $f: S^{2n}\to S^n$ such that $\mu_2$ is decomposed as follows:

$$S^n\times S^n \xrightarrow[\phi]{} (S^n\times S^n)\vee S^{2n} \xrightarrow[1+f]{} S^n$$

where $\phi$ denotes a map pinching to the boundary of a small $2n$-disk imbedded in $S^n\times S^n$ to a point. By Theorem 1 and the above decomposition we have

$$\begin{aligned}\chi_2(\alpha)&=\alpha_*(c(\mu_2))=\alpha_*(\Sigma\mu_1+\Sigma f)(\Sigma\phi)(d(n, n))\\ &=\alpha_*(\Sigma\mu_1+\Sigma f)(d(n, n)+(0))=\alpha_*(\Sigma\mu_1)(d(n, n)).\\ &=\alpha_*(c(\mu_1))=\chi_1(\alpha).\end{aligned}$$

Thus the proof is completed.

**§4. Examples.** If $n$ is even there exists no map of type $(a, b)$ except $(a, 0)$ or $(0, a)$, so we suppose that $n$ is odd in this section. Let $\mu: S^n\times S^n\to S^n$ be a map of type $(a, b)$ (if $n=3, 7, a, b$ are arbitrary and otherwise $ab$ is even), then by Proposition 2 we may assume that

$$\begin{aligned}c(\mu)&=abh_{n+1} \qquad \text{if } n=3,7\\ &=ab/2[\iota_{n+1}, \iota_{n+1}] \qquad \text{otherwise}\end{aligned}$$

where $h_{n+1}$ denotes the Hopf map of $\pi_{2n+1}(S^{n+1})$.

(1) $\pi_{n+1}(S^{n+1})$. We identify an element of this group with an integer $m$ through its degree. Since we have

$$\begin{aligned}m_*(c(\mu))&=abm2h_{n+1} \qquad \text{if } n=3,7\\ &=(ab/2)m2[\iota_{n+1}, \iota_{n+1}] \qquad \text{otherwise.}\end{aligned}$$

We see that an element of $\pi_{n+1}(S^{n+1})$ is $\mu$-homomorphic if and only if it is trivial.

(2) $\pi_{n+1}(S^n)$ $(n\geqq 3)$. This group contains only one non-trivial element $\eta_n$ of order 2. In this case we have

$$m_*(c(\mu)) = ab\eta_n h_{n+1} \qquad \text{if } n=3,7$$
$$= ab/2[\eta_n, \eta_n] \qquad \text{otherwise.}$$

Hence we see that if $n$ is $3, 7$, $\eta_n$ is $\mu$-homomorphic if and only if $ab$ is even, otherewise $\eta_n$ is $\mu$-homomorphic if and only if $ab \equiv 0 \bmod 4$ or $ab \equiv 1 \bmod 4$ and $[\eta_n, \eta_n]=0$ (this depends on $n$).

## References

[1] I. M. James: Reduced product spaces. Ann. of Math., **62**, 170–197 (1955).
[2] ——: On the suspension triad. ibid., **63**, 191–247 (1956).
[3] H. Takahashi: Homomorphisms from $S^3$ to compact Lie groups up to homotopy (to appear in Bull. of Nagaoka Univ. of Tech., no. 11 (1990)).
[4] G. W. Whitehead: Elements of Homotopy Theory. Springer, G.T.M., **61** (1978).