35. *Diophantine Approximations for Periods of Exponential and Elliptic Functions*

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This is to announce the results of the paper [10] which will appear with complete proofs. Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3, associated with a period lattice Ω of $\boldsymbol{C}$. Let $\mathcal{O}$ be the endomorphism ring of $\wp$, that is, the ring of complex numbers ρ such that the lattice $\rho\Omega$ is contained in Ω. We know that $\mathcal{O}$ is either the ring $\boldsymbol{Z}$ of rational integers, or a subring of finite index of the ring of integers of a complex quadratic field k. If $\mathcal{O}\neq \boldsymbol{Z}$, we say that $\wp$ has complex multiplication over k. Let $\omega_1, \omega_2 \in \Omega$ be two periods of $\wp$, which are linearly independent over the field of real numbers $\boldsymbol{R}$, and $\omega \in \Omega$ be a non-zero period of $\wp$.

(a) *Historical survey.* We begin with some history on the transcendence measures concerning with these periods. C.L. Siegel [19] observed in 1932 that the period lattice contains a transcendental number. The transcendence of ω and π/ω follows from a theorem proved by Th. Schneider in 1937 (see for example [18]). If $\wp$ has complex multiplication, the number ω_2/ω_1 belongs to the field of complex multiplication k and Schneider showed the converse, namely, if ω_2/ω_1 is algebraic, then $\wp$ has complex multiplication. Let ζ be the Weierstrass zeta function associated with the same lattice Ω (see for example the definition in [24]). We denote $\eta_1=2\zeta(\omega_1/2)$, $\eta_2=2\zeta(\omega_2/2)$, quasi-periods of ζ. Schneider proved in 1937 that the numbers 1, ω_1, η_1 are lineary independent over $\bar{\boldsymbol{Q}}$ (we denote always by $\bar{\boldsymbol{Q}}$ the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$, namely, the field of all algebraic numbers). See also Schneider's book [18]. Later, A. Baker [2] studied the linear independence of the numbers 1, ω_1, ω_2, η_1, η_2 over $\bar{\boldsymbol{Q}}$, and J. Coates and D.W. Masser looked into (cf. [5] and [12]) the linear independence over $\bar{\boldsymbol{Q}}$ of the 6 numbers 1, ω_1, ω_2, η_1, η_2 and $2\pi i$. The final result (cf. [12]) shows that these 6 numbers are linearly independent over $\bar{\boldsymbol{Q}}$ if $\wp$ has no complex multiplication, while, if $\wp$ has complex multiplication, the dimension of the vector space generated by these 6 numbers over $\bar{\boldsymbol{Q}}$ is 4.

Quantitative refinements of these results are the following. Recall that if β is an algebraic number, we define the usual height $H(\beta)$ as the maximum of the absolute values of the coefficients of the minimal polynomial of β over $\boldsymbol{Z}$. In 1970, Baker obtained in [3] that for $\beta_1, \beta_2 \in \bar{\boldsymbol{Q}}$ of usual height $\leq H$ with $H\geq e$, there exist an absolute constant $\kappa>0$ and an effective constant $c>0$ which depends only on ω_1, ω_2 and the degrees of $\beta_i (i=1, 2)$ such that

$$\log|1+\beta_1\omega_1+\beta_2\omega_2|>-c(\log H)^{\kappa}.$$

N.I. Fel'dman showed in [8] in 1974 a variant of this inequality in the following case. Let D be the degree of an algebraic number β. He obtained that if $\omega_2/\omega_1\neq\beta$ then

$$\log\left|\frac{\omega_2}{\omega_1}-\beta\right|>-c\{D(\log H)^3+D^4(\log D)^3\}, \tag{1.1}$$

where $c>0$ is an effectively calculable constant which depends only on ω_1, ω_2. Independently, a little bit weaker lower bound was shown in 1975 by Masser in [12]. Masser gave also an estimate for a linear combination of 1, ω, $2\pi i$ in algebraic coefficients of height $\leq H$ with $H\geq e^e$ and of degree $\leq D$: for all $\varepsilon>0$, there exists an effective constant $c=c(\varepsilon,\omega_1,\omega_2,D)>0$ such that we have

$$\log|1+\beta_1\omega+\beta_2\cdot 2\pi i|>-c(\log H)(\log\log H)^{4+\varepsilon}.$$

For the transcendence measure of π/ω, E. Reyssat obtained an estimate in [17]

$$\log\left|\frac{\pi}{\omega}-\beta\right|>-c\{(\log H)^2+D^2(\log D)^2\}. \tag{1.2}$$

This lower bound, as well as the estimate (1.1) of Fel'dman, are deduced from a general result on periodic function in [9].

Moreover, let $\wp$, $\wp^*$ be two Weierstrass elliptic functions associated with lattices Ω, Ω^*, respectively. For non-zero periods $\omega\in\Omega$ and $\omega^*\in\Omega^*$, if $\wp(\omega z)$ and $\wp^*(\omega^* z)$ are algebraically independent over $\boldsymbol{C}$, a result of W.D. Brownawell and Masser in [4] shows that

$$\log\left|\frac{\omega^*}{\omega}-\beta\right|>-c\{(\log H)^3+D^8(\log D)^3\}. \tag{1.3}$$

Now, in our paper, we give some estimates on transcendence measures, for the quotient of two periods of $\wp$, or for the quotient of one period of $\wp$ and $2\pi i$, or further, for the quotient of one period of $\wp$ and one period of $\wp^*$. A more general lower bound for linear forms in algebraic points of the exponential map of commutative algebraic groups will be treated in another paper of the author's [11], that improves the estimate of Philippon and Waldschmidt [15].

The new idea which is essential to improve previous results is a kind of generalizations of a technique due to Fel'dman, Masser and Reyssat. This is based on the fact that the derivative of order $>n$ of z^n vanishes. Fel'dman used this idea in [7] to obtain transcendence measures of the numbers $\log\alpha,\pi,\omega$ and u, where α is an algebraic number $\neq 0$, $\neq 1$ and u denotes an algebraic point of $\wp$ (see for example the definition in [1]). Masser applied this technique in [12] to get a lower bound for $\beta_0+\beta_1\omega+\beta_2\cdot 2\pi i$ under the hypothesis $\beta_0\neq 0$. The refinement by Reyssat for the transcendence measures of ω and u is also due to this technique in [17]. We explain here what was difficult in order to extend their works. A priori, this idea needs to assume $\beta_0 \neq 0$ for example to get an estimate for $\beta_0+\beta_1\omega+\beta_2\cdot 2\pi i$. Our method to treat the case $\beta_0=0$ consists in writing $\beta_1\omega+\beta_2\cdot 2\pi i=\beta_0'.\ 0+\beta_1\omega+\beta_2\cdot 2\pi i$ with $\beta_0'\neq 0$. This allows us to apply the technique to the case of homogeneous linear forms. Secondly, our auxiliary function F is, for example, a polynomial in z_0, $\exp(z_1)$ and $\wp(z_2+\omega_0/2)$ for the proof of Theorem 1.5 and we give an estimate of the values of F at the points $(0, 2si\pi, s\omega)$ for integers s. We see these points are periods of F. Baker's method traditionally permits to extrapolate on these points; however, that is useless in the case of periods. We have to extrapolate on the derivatives, and it is possible by a method in [22].

(b) *Two main theorems.* To announce the precise results, let us recall the definition of Weil's logarithmic absolute height h: for $\alpha=(\alpha_0, \cdots, \alpha_N) \in \boldsymbol{P}_N(\bar{\boldsymbol{Q}})$, if K is a number field which contains $\alpha_0, \cdots, \alpha_N$, we define h by

$$h(\alpha)=\frac{1}{[K:\boldsymbol{Q}]}\sum_{\nu}[K_\nu:\boldsymbol{Q}_\nu]\cdot\log \operatorname{Max}\{|\alpha_j|_\nu\,;\,0\le j\le N\},$$

where ν runs over the set of all places of K and $[K_\nu:\boldsymbol{Q}_\nu]$ is the local degree such that the product formula is written down in the from

$$\sum_{\nu}[K_\nu:\boldsymbol{Q}_\nu]\cdot\log|\gamma|_\nu=0, \qquad \text{where } \gamma\in K,\ \gamma\neq 0.$$

For $\beta\in\bar{\boldsymbol{Q}}$, we denote by $h(\beta)$ Weil's logarithmic absolute height at the point $(1, \beta)\in \boldsymbol{P}_1(\bar{\boldsymbol{Q}})$, The relation between $h(\beta)$ and the usual height $H(\beta)$ is explained in Chapter 1 in [20]. For example, we have

$$h(\beta)\le\frac{\log H(\beta)+\log D}{D} \qquad \text{for } \beta\in\bar{\boldsymbol{Q}}, \tag{1.4}$$

where D denotes the degree of β over $\boldsymbol{Q}$.

Now, we announce the following results.

Theorem 1.5. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let ω be a non-zero period of $\wp$. There exists an effective constant $c>0$ depending only on the heights of g_2, g_3, the degrees of g_2, g_3 and the number $|\omega|$ with the following properties: let β_1, β_2 be two non-zero algebraic numbers such that $[\boldsymbol{Q}(\beta_1, \beta_2):\boldsymbol{Q}]\le D$ and B be a real number satisfying*

$$\log B\ge \operatorname{Max}(e, h(\beta_1), h(\beta_2)).$$

We put $\Lambda=\beta_1\cdot 2\pi i+\beta_2\omega$. Then we have

$$\log|\Lambda|>-c\cdot D^2(\log B+\log D)(\log\log B+\log D). \tag{1.6}$$

Theorem 1.7. *Let $\wp_1$ (resp. $\wp_2$) be a Weierstrass elliptic function with algebraic invariants g_{21}, g_{31} (resp. g_{22}, g_{32}). Let ω_1 (resp. ω_2) be a non-zero period of $\wp_1$ (resp. $\wp_2$). There exists an effective constant $c>0$ depending only on the heights of $g_{21}, g_{31}, g_{22}, g_{32}$, the degrees of $g_{21}, g_{31}, g_{22}, g_{32}$ and the numbers $|\omega_1|$, $|\omega_2|$ with the following properties: let β_1, β_2 be two non-zero algebraic numbers such that $[\boldsymbol{Q}(\beta_1, \beta_2):\boldsymbol{Q}]\le D$ and B be a real number satisfying*

$$\log B\ge \operatorname{Max}(e, h(\beta_1), h(\beta_2)).$$

We put $\Lambda=\beta_1\omega_1+\beta_2\omega_2$. If $\Lambda\neq 0$, then we have

$$\log|\Lambda|>-c\cdot D^3(\log B+\log D)(\log\log B+\log D)^2. \tag{1.8}$$

(c) *A corollary.* We can apply Theorem 1.7 to the quotient of two periods of an elliptic function.

Corollary 1.9. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let ω_1, ω_2 be two periods of $\wp$, which are linearly independent over* $\boldsymbol{R}$. *We suppose that $\wp$ has no complex multiplication. There exists an effective constant $c>0$ depending only on the heights of g_2, g_3, the degrees of g_2, g_3 and the absolute values $|\omega_1|$, $|\omega_2|$ with the following properties: let β_1, β_2 be two non-zero algebraic numbers such that $[\boldsymbol{Q}(\beta_1, \beta_2): \boldsymbol{Q}]\leq D$ and B be a real number satisfying*

$$\log B \geq \mathrm{Max}\,(e, h(\beta_1), h(\beta_2)).$$

We put $\Lambda=\beta_1\omega_1+\beta_2\omega_2$. Then we have

$$\log|\Lambda| > -c\cdot D^3(\log B+\log D)(\log\log B+\log D)^2. \tag{1.10}$$

(d) *Three transcendence measures.* The three previous announcements can be formulated in terms of the usual height. Then they give transcendence measures. Let us recall that the usual height of a polynomial is the maximum of the absolute values of its coefficients.

Corollary 1.11. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let ω_1, ω_2 be two periods of $\wp$, which are linearly independent over* $\boldsymbol{R}$. *We suppose that $\wp$ has no complex multiplication. There exists an effective constant $c>0$ depending only on the heights of g_2, g_3, the degrees of g_2, g_3 and the absolute values $|\omega_1|$, $|\omega_2|$ with the following properties: let $P\in \boldsymbol{Z}[X]$ be a non-zero polynomial of degree $\leq D$ and of height $\leq H$ with $H\geq e^e$. Then we have*

$$\log\left|P\left(\frac{\omega_2}{\omega_1}\right)\right| > -c\{D^2\log H(\log\log H)^2+D^3(\log D)^3\}. \tag{1.12}$$

Remark. We know that when $\wp$ has complex multiplication, the number ω_2/ω_1 is algebraic, then inequalities (1.10) and (1.12) are trivial because of Liouville's inequality (see for example Lemme 3 in [13]). By the same reason, in Theorem 1.7, when the two functions $\wp_1(\omega_1 z)$ and $\wp_2(\omega_2 z)$ are algebraically dependent, then the quotient ω_2/ω_1 is algebraic and inequality (1.8) is trivial.

We recall that Fel'dman's result (1.1) was the best known estimate [8] until now. Lower bound (1.12) improves that of Fel'dman as well as that of [9].

Corollary 1.13. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let ω be a non-zero period of $\wp$. There exists an effective constant $c>0$ depending only on the heights of g_2, g_3, the degrees of g_2, g_3 and the absolute value $|\omega|$ with the following properties: let $P\in\boldsymbol{Z}[X]$ be a non-zero polynomial of degree $\leq D$ and of height $\leq H$ with $H\geq e^e$. Then we have*

$$\log\left|P\left(\frac{\pi}{\omega}\right)\right| > -c\{D\log H\cdot\log\log H+D^2(\log D)^2\}. \tag{1.14}$$

Estimate (1.14) is better than that of Reyssat (1.2) bacause we can see easily

$$D\log H\cdot\log\log H < c'\{(\log H)^2+D^2(\log D)^2\}.$$

Corollary 1.15. *Let $\wp_1$ (resp. $\wp_2$) be a Weierstrass elliptic function with algebraic invariants g_{21}, g_{31} (resp. g_{22}, g_{32}). Let ω_1 (resp. ω_2) be a non-zero period of $\wp_1$ (resp. $\wp_2$). We suppose that the two functions $\wp_1(\omega_1 z)$ and $\wp_2(\omega_2 z)$ are algebraically independent. There exists an effective constant $c>0$ depending only on the heights of $g_{21}, g_{31}, g_{22}, g_{32}$, the degrees of $g_{21}, g_{31}, g_{22}, g_{32}$ and the absolute values $|\omega_1|, |\omega_2|$ with the following properties: let $P \in Z[X]$ be a non-zero polynomial of degree $\leq D$, of height $\leq H$ with $H \geq e^e$. Then we have*

$$\log\left|P\left(\frac{\omega_2}{\omega_1}\right)\right| > -c\{D^2 \log H(\log\log H)^2 + D^3(\log D)^3\}. \tag{1.16}$$

Bound (1.16) is better than (1.3), i.e., that of Brownawell and Masser [4].

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