27. Spectral Properties of the Operator Associated with a Retarded Functional Differential Equation in Hilbert Space

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In [4] the fundamental result on the structural operator for the linear retarded functional differential equation

(1) $$du(t)/dt=A_0u(t)+A_1u(t-h)+\int_{-h}^{0}a(s)A_2u(t+s)ds$$

in a Hilbert space H was established. Here, $-A_0$ is the operator associated with a bounded sesquilinear form $a(u,v)$ defined in $V\times V$ and satisfying Gårding's inequality

$$\operatorname{Re} a(u,u)\geq c\|u\|^2, \quad c>0,$$

where V is a Hilbert space densely and continuously imbedded in H and $\|\ \|$ is the norm of V. It is known that A_0 generates an analytic semigroup in both of H and V^*. It is assumed that A_1 and A_2 are bounded linear operators from V to V^* and $A_iA_0^{-1}$, $i=1, 2$, are bounded also in H. The real valued function $a(s)$ is assumed to be Hölder continuous in $[-h, 0]$.

Let $S(t)$: $M=H\times L^2(-h,0;V)\to M$ be the solution semigroup for (1) considered as an equation in V^*: for $g=(g^0, g^1)\in M$

$$S(t)g=(u(t;g),\quad u(t+\cdot\,;g)),$$

where $u(t;g)$ is the mild solution of (1) satisfying the initial conditions

(2) $$u(0;g)=g^0,\quad u(s;g)=g^1(s) \qquad \text{for } s\in[-h,0).$$

In this paper we investigate the spectral properties of the infinitesimal generator A of $S(t)$ in the special case where $A_1=\gamma A_0$ with some real constant γ, $A_2=A_0$ and the imbedding $V\subset H$ is compact. Hence, in what follows throughout this paper we consider the equation

(3) $$du(t)/dt=A_0u(t)+\gamma A_0u(t-h)+\int_{-h}^{0}a(s)A_0u(t+s)ds$$

with A_0, γ, a satisfying the assumptions stated above.

According to the Riesz-Schauder theory A_0 has a discrete spectrum: $\sigma(A_0)=\{\mu_j:\ j=1,2,\cdots\}$. Set

(4) $$m(\lambda)=1+\gamma e^{-\lambda h}+\int_{-h}^{0}e^{\lambda s}a(s)ds.$$

It is clear that $m(\lambda)$ is an entire function and

(5) $$m(\lambda)\to 1 \qquad \text{as } \operatorname{Re}\lambda\to+\infty.$$

The following lemmas are proved as Theorems 6.1 and 7.2 of

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S. Nakagiri [2].

Lemma 1. *$(\lambda-A)f=\phi$ if ond anly if*

$$\Delta(\lambda)f^0=\phi^0+\int_{-h}^0 e^{-\lambda(h+\tau)}\gamma A_0\phi^1(\tau)d\tau+\int_{-h}^0 a(s)\int_s^0 e^{\lambda(s-\tau)}A_0\phi^1(\tau)d\tau ds,$$

$$f^1(s)=e^{\lambda s}f^0+\int_s^0 e^{\lambda(s-\tau)}\phi^1(\tau)d\tau,$$

where $\Delta(\lambda)=\lambda-m(\lambda)A_0$.

Lemma 2. *For $l=1, 2, \cdots$*

$$\mathrm{Ker}(\lambda-A)^l=\Big\{\Big(\phi_0^0, e^{\lambda s}\sum_{i=0}^{l-1}(-s)^i\phi_i^0/i!\Big): \sum_{i=j-1}^{l-1}(-1)^{i-j}\Delta^{(i-j+1)}(\lambda)\phi_i^0/(i-j+1)!=0,\ j=1, \cdots, l\Big\}.$$

Theorem 1. *Let $\sigma(A)$ be the spectrum of the infinitesimal generator A of $S(t)$. Then*

$$\sigma(A)=\sigma_e(A)\cup\sigma_p(A),$$

where $\sigma_e(A)=\{\lambda: m(\lambda)=0\}$ and $\sigma_p(A)=\{\lambda: m(\lambda)\neq 0, \lambda/m(\lambda)\in\sigma(A_0)\}$. Each nonzero point of $\sigma_e(A)$ is not an eigenvalue of A and is a cluster point of $\sigma(A)$. $\sigma_p(A)$ consists only of discrete eigenvalues.

Suppose $m(0)=0$. Then, 0 is an eigenvalue of A with infinite multiplicity. 0 is an isolated point of $\sigma(A)$ if it is a simple zero of $m(\lambda)$, and is a cluster point of $\sigma(A)$ if it is a multiple zero of $m(\lambda)$.

Outline of the proof. With the aid of the Riesz-Schauder theory and Lemma 1 it is not difficult to verify that

$$\rho(A)=\{\lambda: m(\lambda)\neq 0, \lambda/m(\lambda)\in\rho(A_0)\}.$$

Suppose $\lambda_0\neq 0$ is a zero of $m(\lambda)$ of the k-th order. Then, there exists a function $h(\lambda)$ which is holomorphic in a neighbourhood of λ_0 such that $m(\lambda)/\lambda=(\lambda-\lambda_0)^k h(\lambda)^k$. Applying the inverse function theorem to $(\lambda-\lambda_0)h(\lambda)$ and noting $\mu_j\to\infty$ we see that for sufficiently large j there exists a complex number λ_j such that $(\lambda_j-\lambda_0)h(\lambda_j)=\mu_j^{-1/k}$ and $\lambda_j\to\lambda_0$. Then, $\lambda_j/m(\lambda_j)=\mu_j\in\sigma(A_0)$, and hence λ_0 is a cluster point of $\sigma(A)$.

Next, suppose that $m(\lambda_0)\neq 0$, $\lambda_0/m(\lambda_0)\in\sigma(A_0)$. If there exists a sequence $\{\lambda_j\}$ such that $\lambda_0\neq\lambda_j\in\sigma(A)$, $m(\lambda_j)\neq 0$ and $\lambda_j\to\lambda_0$, then $\lambda_j/m(\lambda_j)\to\lambda_0/m(\lambda_0)$, $\lambda_j/m(\lambda_j)\in\sigma(A_0)$. Since $\sigma(A_0)$ consists only of isolated points, we have $\lambda_j/m(\lambda_j)=\lambda_0/m(\lambda_0)$ for sufficiently large j. In view of the theorem of identity we have $\lambda/m(\lambda)\equiv\lambda_0/m(\lambda_0)$ which contradicts (5).

Theorem 2. *Suppose that $m(0)\neq 0$, $\gamma\neq 0$ and the generalized eigenvectors of A_0 are complete in H. Then, the generalized eigenvectors of A are complete in M.*

Outline of the proof. Let P_n be the spectral projection to the generalized eigenspace of A_0 associated with $\mu_n\in\sigma(A_0)$. Set $H_n=P_nH$ and $A_{0_n}=A_0|_{H_n}$. Then, clearly $P_nV=H_n$. If we denote the solution semigroup of the equation (3) with A_{0_n} in place of A_0 by $S_n(t)=\exp(tA_n)$, then the commutativity of A_0 and P_n yields

$$S_n(t)=S(t)|_{M_n} \quad\text{and}\quad A_n=A|_{D(A_n)},$$

where $M_n=H_n\times L^2(-h, 0; H_n)$. It follows from Lemma 2 that for λ with

$\lambda/m(\lambda)=\mu_n$, $(\lambda-A)^l\phi=0$ if and only if $(\lambda-A_n)^l\phi=0$. Thus, the assertion of the theorem follows from the corresponding result of A. Manitius ([1]: Theorems 5.1 and 5.4(ii)) in the case of a finite dimensional space.

As an application we consider the identification problems for the equation (3) (cf. Theorem 3.1 of S. Nakagiri and M. Yamamato [3]).

We denote by (3^m) the equation (3) with A_0, γ, a replaced by A_0^m, γ^m, a^m respectively. The mild solution of (3^m) satisfying the initial conditions (2) is denoted by $u^m(t;g)$, and the solution semigroup for (3^m) by $S^m(t)=\exp(tA^m)$. We assume that A_0^m and a^m satisfy the same type of assumptions as A_0 and a. Hence, the conclusion of Theorem 1 holds also for A^m.

Let $g^i=(g_i^0, g_i^1)\in M$, $i=1,\cdots,q$, be a finite set of initial values. We say that A_0, γ, a are identifiable if $A_0=A_0^m$, $\gamma=\gamma^m$, $a(s)\equiv a^m(s)$ follows from

$$u(t;g^i)\equiv u^m(t;g^i),\quad i=1,\cdots,q.$$

Let $\{\mu_n^m: n=1,2,\cdots\}$ be the set of eigenvalues of A_0^m, and by $\{\psi_{n1}^0,\cdots,\psi_{nd_n}^0\}$ a base of $\mathrm{Ker}(\overline{\mu_n^m}-(A_0^m)^*)$, where $d_n=\dim\mathrm{Ker}(\mu_n^m-A_0^m)$. Let $\{\lambda_{nj}^m: j=1,2,\cdots\}$ be the totality of the complex numbers λ satisfying $\lambda/m^m(\lambda)=\mu_n^m$. If we set $\psi_{nj}^k=(\psi_{nk}^0, \exp(\overline{\lambda_{nj}^m}s)\psi_{nk}^0)$, then $\{\psi_{nj}^k: k=1,\cdots,d_n\}$ is a base of $\mathrm{Ker}(\overline{\lambda_{nj}^m}-A_T^m)$, where A_T^m is the infinitesimal generator of the solution semigroup associated with the equation (3^m) with A_0^m replaced by its adjoint $(A_0^m)^*$.

By F^m we denote the structural operator for (3^m) and by $(\ ,\ )_M$ the duality between M^* and M.

Theorem 3. *Suppose that $\gamma^m\neq 0$, $m^m(0)\neq 0$ and the generalized eigenvectors of A_0^m are complete in H. If the set of initial values $\{g^1,\cdots,g^q\}$ satisfies*

$$\mathrm{rank}\left((F^m g^i, \psi_{nj}^k)_M:\ \begin{matrix} i\rightarrow 1,\cdots,q \\ k\downarrow 1,\cdots,d_n\end{matrix}\right)=d_n$$

for each n, j, then A_0, γ, a are identifiable.

One can prove this theorem following the proof of Proposition 3.1 and Theorem 3.1 of [3] and taking Remark 3.2 of the same paper into consideration. The only difference is to show $\sigma_e(A^m)\subset\sigma_e(A)$ and $\sigma_p(A^m)\subset\sigma_p(A)$ instead of $\sigma(A^m)\subset\sigma(A)$ to start with.

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