22. *Convex Operators and Convex Integrands*

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Introduction. Let Ω be an arbitrary set, let Σ be a σ-field of subsets of Ω (the measurable sets), and let μ denote a nonnegative σ-finite measure on Σ. Let $S(\Omega)$ be the space of all finite valued measurable functions on Ω. We identify f and $g \in S(\Omega)$ if they differ only on a set of μ-measure zero. With the usual ordering $S(\Omega)$ is an order complete vector lattice. A mapping $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ is called a convex operator if $D(F)$ the domain of F is a convex set of the d-dimensional Euclidean space $\boldsymbol{R}^d$ and

$$F(\lambda x+(1-\lambda)y) \leqq \lambda F(x)+(1-\lambda)F(y)$$

holds for every $x, y \in D(F)$ and $\lambda \in [0, 1]$. Many kinds of ordered vector spaces can be regarded as the subspaces of $S(\Omega)$, and hence this class of convex operators covers many cases. A function $f: \boldsymbol{R}^d \times \Omega \to \boldsymbol{R} \cup \{+\infty\}$ is said to be a *convex integrand* if $f(\cdot, t)$ is a convex function for each $t \in \Omega$. We say that a convex integrand f is a *representation* of a convex operator F if $f(x, t)$ is measurable in $t \in \Omega$ and $f(x, \cdot)=F(x)$ holds for every $x \in D(F)$. Our main result (Theorem 1) asserts that every convex operator $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ has at least a representation of F. In [3], one can see the proof of this result in one dimensional case. In general case, the proof is more complicated. In § 2, we consider the relations between convex operators and their representations. In § 3, we generalize the Fenchel-Moreau theorem by using representations, and give some conditions with which a convex operator can be represented by a normal convex integrand.

§ 1. Representation theorem.

Theorem 1. *For every convex operator $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$, there exists at least a representation of F.*

Outline of the proof. The proop is done by constructing a representation. The difficulty is to determine the value of $f(x, t)$ when x belongs to $\partial D(F)$ the boundary of $D(F)$. For each $x \in \partial D(F)$, let L_x be the largest linear manifold such that some neighborhoods of x in L_x are contained by $\partial D(F)$. First we define the values of $f(x, t)$ on $D^\circ(F) \times \Omega$ by a countable argument which is an analogy of the proof in one dimensional case. Next, for each L_x with $\dim L_x = d-1$, we define $f(y, t)$ on $(L_x \cap \partial D(F)) \times \Omega$ satisfying the followings.

(a) $\sup_{z \in D(F)} \lim_{\lambda \to 0} f(y+\lambda(z-y), t) \leqq f(y, t)$ for every $y \in L_x$,

(b) $f(\cdot, t)$ is convex on L_x on $L_x \cap \partial D(F)$ for every $t \in \Omega$,

(c) $f(y, \cdot)=(F(y))(\cdot)$ for every $y \in L_x \cap D(F)$.

We can choose such values for $f(y, t)$ if we use the fact that, for each $y \in L_x$,

$$\sup_{z\in D(F)}\lim_{\lambda\to 0} f(y+\lambda(z-y),t)\leqq (F(y))(t)$$

holds for almost every $t\in\Omega$. Next, for each L_x with $\dim L_x=d-2$, we define $f(y,t)$ on $(L_x\cap\partial D(F))\times\Omega$ in the similar way. By this iteration, $f(y,t)$ will be defined on $\overline{D(F)}\times\Omega$. Finally, by defining $f(y,t)=\infty$ for every exterior point y of $D(F)$ and $t\in\Omega$, we complete the construction of the representation of F.

§2. **Normal representations.** A convex integrand $f\colon \boldsymbol{R}^d\times\Omega\to\boldsymbol{R}\cup\{+\infty\}$ is said to be *normal* if $f(\cdot,t)$ is lower semicontinuous for every $t\in\Omega$ and there exists a countable family of measurable functions $\xi_n\colon\Omega\to\boldsymbol{R}^d$ $(n=1,2,\cdots)$ such that

(1) $f(\xi_n(t),t)$ is measurable in $t\in\Omega$ for each n and

(2) $\{\xi_n(t)\}_{n=1}^{\infty}$ is dense in $D(f(\cdot,t))$ for each $t\in\Omega$,

where $D(f(\cdot,t))=\{x\in\boldsymbol{R}^d\,|\,f(x,t)<+\infty\}$. If a convex integrand f is normal, then $f(\xi(t),t)$ is measurable in $t\in\Omega$ whenever $\xi\colon\Omega\to\boldsymbol{R}^d$ is measurable. (See [6].) We say that a convex operator F has a *normal representation* if there exists a normal convex integrand f which represents F. In Theorem 1, we note that the existence of a representation is not unique and that every convex operator does not have a normal representation. We will find some conditions with which a convex operator has a normal representation. By the conjugate of convex integrand f, we shall mean the convex integrand $f^*\colon\boldsymbol{R}^d\to\boldsymbol{R}\cup\{+\infty\}$ defined by

$$f^*(\xi,t)=\sup_{x\in\boldsymbol{R}^d}\{\langle x,\xi\rangle-f(x,t)\}.$$

Also the biconjugate integrand $f^{**}\colon\boldsymbol{R}^d\times\Omega\to\boldsymbol{R}\cup\{+\infty\}$ is given by

$$f^{**}(x,t)=\sup_{\xi\in\boldsymbol{R}^d}\{\langle x,\xi\rangle-f^*(\xi,t)\}.$$

If a convex integrand f is normal, then so are f^* and f^{**}. We note that for a convex operator F we can choose a representation f of F such that $f(x,t)=\infty$ if $x\notin\overline{D(F)}$, that is $\overline{D(f(\cdot,t))}$ does not depend on $t\in\Omega$.

Lemma 1. *Let $f\colon\boldsymbol{R}^d\times\Omega\to\boldsymbol{R}\cup\{+\infty\}$ be a representation such that $\overline{D(f(\cdot,t))}$ does not depend on $t\in\Omega$. Then f is normal if and only if $f(\cdot,t)$ is lower semicontinuous, in other words, $f^{**}=f$ on $\boldsymbol{R}^d\times\Omega$.*

Proof. (See [4].)

Next, we define the conjugate of a convex operator $F\colon\boldsymbol{R}^d\supset D(F)\to S(\Omega)$. Let $L(\boldsymbol{R}^d,S(\Omega))$ denote the space of all linear mapping from $\boldsymbol{R}^d$ to $S(\Omega)$. We identify $L(\boldsymbol{R}^d,S(\Omega))$ with the set $S(\Omega)^d=\{\xi=(\xi_1,\cdots,\xi_d)\,|\,\xi_i\in S(\Omega)\ i=1,\cdots,d\}$ by corresponding $S(\Omega)^d\ni(\xi_1,\cdots,\xi_d)$ to $\phi\colon\boldsymbol{R}^d\ni(x_1,\cdots,x_d)\to\langle x,\xi\rangle=x_1\xi_1+\cdots+x_d\xi_d S(\Omega)$. For a convex operator $F\colon\boldsymbol{R}^d\supset D(F)\to S(\Omega)$, the conjugate operator $F^*\colon L(\boldsymbol{R}^d,S(\Omega))\supset D(F^*)\to S(\Omega)$ is defined by

$$F^*(\xi)=\bigvee_{x\in D(F^*)}(\langle x,\xi\rangle-F(x))$$

where $\vee$ means the lattice supremum in the space $S(\Omega)$, and $D(F^*)$ is the set of all $\xi\in S(\Omega)^d$ such that the supremum $F^*(\xi)$ exists. The biconjugate operator F^{**} is defined on the space $L(L(\boldsymbol{R}^d,S(\Omega)),S(\Omega))=L(S(\Omega)^d,S(\Omega))$, and we regard $S(\Omega)^d$ and $\boldsymbol{R}^d$ as the subspaces of this by corresponding $\eta\in$

$S(\Omega)^d$ and $x \in \boldsymbol{R}^d$ to $\langle \eta, \cdot \rangle$ and $\langle x, \cdot \rangle \in L(S(\Omega)^d, S(\Omega))$ respectively. For $x \in \boldsymbol{R}^d$ and $\eta \in S(\Omega)^d$, F^{**} is defind by

$$F^{**}(x) = \bigvee_{\xi \in D(F^*)} (\langle x, \xi \rangle - F^*(\xi))$$

$$F^{**}(\eta) = \bigvee_{\xi \in D(F^*)} (\langle \eta, \xi \rangle - F^*(\xi)).$$

They are only defined on the domain $D(F^{**})$ where these suprema exist.

Theorem 2. *Let $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ be a convex operator, and let $f: \boldsymbol{R}^d \times \Omega \to \boldsymbol{R} \cup \{+\infty\}$ be a representation of F. Then the convex integrands f^* and f^{**} are normal representations of F^* and F^{**} respectively. Moreover for $\xi \in D(F^*)$ and $\eta \in D(F^{**})$,*

$$(F^*(\xi))(t) = f^*(\xi(t), t)$$

$$(F^{**}(\eta))(t) = f^{**}(\eta(t), t)$$

holds for almost every $t \in \Omega$.

The proof of this theorem is not complicated if we use the following fundamental lemma.

Lemma 2. *Let $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ be a convex operator, and let $f: \boldsymbol{R}^d \times \Omega \to \boldsymbol{R} \cup \{+\infty\}$ be a representation of F. Let U be a convex subset of $D(F)$ and suppose that $\inf_{x \in U} f(x, t) > -\infty$ for almost every $t \in \Omega$. Then $\bigwedge_{x \in U} F(x) \in S(\Omega)$ exists and*

$$(\bigvee_{x \in U} F(x))(t) = \inf_{x \in U} f(x, t).$$

Combining Lemma 1 and Theorem 2, we obtain the following result.

Theorem 3. *A convex operator $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ has a normal representation if and only if $F^{**}(x) = F(x)$ holds for every $x \in D(F)$.*

§3. A generalization of Fenchel-Moreau theorem. In this section, we give a definition which can be regarded as a generalization of the notion of lower semicontinuity of convex operators. For a convex operator $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$, and for $z \in D(F)$, we denote

$$S_F(z) = \{\phi \in S(\Omega)^+ \mid F(U \cap D(F)) \subset F(z) - y + S(\Omega)^+ \text{ for some neighborhood } U \text{ of } z\}$$

where

$$S(\Omega)^+ = \{\phi \in S(\Omega) \mid \phi(t) \geqq 0 \text{ for almost every } t \in \Omega\}.$$

Lemma 3. *Let $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ be a convex operator, let $f: \boldsymbol{R}^d \times \Omega \to R \cup \{+\infty\}$ be a representation of F, and take a point $x \in D(F)$. If for almost every $t \in \Omega$ the convex functions $f(\cdot, t)$ are lower semicontinuous at x, then $S_F(x) \neq \phi$ and $\wedge S_F(x) = 0$.*

Now we can give a generalization of Fenchel-Moreau theorem.

Theorem 4. *Let $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ be a convex operator, and let x be a point of $D(F)$. Then $F^{**}(x) = F(x)$ if and only if $S_F(x) \neq \phi$ and $\wedge S_F(x) = 0$.*

Proof. Let $f: \boldsymbol{R}^d \times \Omega \to R \cup \{+\infty\}$ be a representation of F, and suppose that $F^{**}(x) = F(x)$. By Theorem 2, $f^{**}(x, t) = f(x, t)$ holds for almost every $t \in \Omega$, and this implies that $f(\cdot, t)$ is lower semicontinuous at x. Hence we see by Lemma 3 that $S_F(x) \neq \phi$ and $\wedge S_F(x) = 0$. The proof of sufficiency is found in [2].

Now, combining Theorem 3 and Theorem 4, we obtain the following result.

Corollary 1. *A convex operator* $F: \boldsymbol{R}^d \supset D(F) \rightarrow S(\Omega)$ *has a normal representation if and only if* $S_F(x) \neq \phi$ *and* $\wedge S_F(x) = 0$ *for all* $x \in D(F)$.

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