49. Compactness Criteria for an Operator Constraint in the Arkin-Levin Variational Problem

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(Communicated by Shokichi IYANAGA, M. J. A., June 13, 1989)

1. Introduction. Let $(S, \mathcal{E}_S, \mu)$ and $(T, \mathcal{E}_T, \nu)$ be measure spaces and assume that a trio of functions $u: S\times T\times \boldsymbol{R}^l \to \boldsymbol{R}$, $g: S\times T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}^k$, and $\omega: T\to \boldsymbol{R}^k$ is given. Consider the well-known Arkin-Levin variational problem formulated as follows:

$$(\boldsymbol{P}) \qquad \begin{aligned} &\mathop{Maximize}_{x} \int_{S\times T} u(s,t,x(s,t))\,d(\mu\otimes\nu) \\ &\text{subject to} \\ &\int_S g(s,t,x(s,t))\,d\mu \leqq \omega(t) \quad a.e. \end{aligned}$$

The existence of optimal solutions for $(\boldsymbol{P})$ has been investigated by Arkin-Levin [1] and Maruyama [5], [6], where a special kind of infinite dimensional Ljapunov measure played a crucial role. In this paper, we shall present a more classical alternative approach to the existence problem, based upon the Continuity Theorem for nonlinear integral functionals due to Ioffe [3] and the Compactness Theorem stated and proved in the next section.

2. Compactness Theorem.

Theorem 1 (Compactness Theorem). *Let $(S, \mathcal{E}_S, \mu)$ and $(T, \mathcal{E}_T, \nu)$ be finite measure spaces and $f: S\times T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}$ be $(\mathcal{E}_S\otimes\mathcal{E}_T\otimes\mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\bar{\boldsymbol{R}}))$-measurable, where $\mathcal{B}(\cdot)$ stands for the Borel σ-field on $(\cdot)$. We denote by $f^*(s,t,\cdot)$ the Young-Fenchel transform of $x\mapsto f(s,t,x)$ for any fixed $(s,t)\in S\times T$; i.e. $f^*(s,t,y)=\sup_x(\langle y,x\rangle - f(s,t,x))$, $y\in \boldsymbol{R}^l$. If f satisfies the growth condition:*

$$\mathrm{Dom}\int_{S\times T} |f^*(s,t,y)|\,d(\mu\otimes\nu) = \boldsymbol{R}^l;$$

$$i.e. \int_{S\times T} |f^*(s,t,y)|\,d(\mu\otimes\nu) < \infty \quad \text{for all } y\in \boldsymbol{R}^l,$$

then the set

$$F_c = \left\{ x\in L^1(S\times T, \boldsymbol{R}^l) \,\middle|\, \int_S f(s,t,x(s,t))\,d\mu \leqq c(t) \;\; a.e. \right\}$$

is weakly relatively compact in $L^1(S\times T, \boldsymbol{R}^l)$ for any $c\in L^1(T,\boldsymbol{R})$.

We need a lemma due to Ioffe-Tihomirov [4] (p. 358–359).

Lemma. *Let $(T, \mathcal{E}, \eta)$ be a measure space and $f: T\times \boldsymbol{R}^l\to \bar{\boldsymbol{R}}$ be a measurable function which satisfies the growth condition:*

$$\mathrm{Dom}\int_T |f^*(t,y)|\,d\eta = \boldsymbol{R}^l; \quad i.e. \int_T |f^*(t,y)|\,d\eta < \infty \quad \text{for all } y\in \boldsymbol{R}^l.$$

And define the function $r_M : T \to \bar{\boldsymbol{R}} (M \geqq 0)$ *by*

$$r_M(t) = \sup_{\|y\| \leqq M} f^*(t, y).$$

Then the function $\theta_c : \boldsymbol{R}_+ \to \bar{\boldsymbol{R}}$ (*for fixed* $c \in \boldsymbol{R}$) *defined by*

$$\theta_c(\tau) = \sup\left\{\inf_{M>0} \frac{1}{M}\left(|c| + \int_E |r_M(t)| d\eta\right) \middle| E \in \mathcal{E}, \eta(E) \leqq \tau\right\}$$

satisfies the following four conditions:

(i) $\theta_c(\tau) \geqq 0$, *for all* $\tau \in \boldsymbol{R}_+$,

(ii) θ_c *is nondecreasing*,

(iii) $\theta_c(0) = 0$, *and*

(iv) $\theta_c(\tau) \to 0$ *as* $\tau \to 0$.

Proof of the Compactness Theorem. We shall show the uniform integrability of F_c through a reasoning analogous to Ioffe-Tihomirov [4] (p. 360–361). By the growth condition of f, we must have

(1) $$\int_S |f^*(s, t, 0)| d\mu = \int_S |-\inf_x f(s, t, x)| d\mu < \infty \quad \text{for } a.e.\ t.$$

If we define the function $c_1 : T \to \boldsymbol{R}$ by

$$c_1(t) = c(t) + \int_S |f^*(s, t, 0)| d\mu,$$

then $c_1 \in L^1(T, \boldsymbol{R})$ because of the growth condition.

It can easily be verified that

(2) $$\int_{E_t} f(s, t, x(s, t)) d\mu \leqq c_1(t) \qquad \text{for } a.e.\ t$$

for any $E \in \mathcal{E}_S \otimes \mathcal{E}_T$ and any $x \in F_c$, where E_t is the section of E at t. The inequality (2) comes from a simple calculation as follows:

(3) $$\begin{aligned} c(t) &\geqq \int_{E_t} f(s, t, x(s, t)) d\mu + \int_{S \setminus E_t} f(s, t, x(s, t)) d\mu \\ &\geqq \int_{E_t} f(s, t, x(s, t)) d\mu - \int_{S \setminus E_t} f^*(s, t, 0) d\mu \\ &\geqq \int_{E_t} f(s, t, x(s, t)) d\mu - \int_S |f^*(s, t, 0)| d\mu. \end{aligned}$$

Integrating the both sides of (3) with respect to t, we obtain

$$\int_E f(s, t, x(s, t)) d(\mu \otimes \nu) \leqq \|c_1\|_1 < \infty.$$

Define $r_M : S \times T \to \bar{\boldsymbol{R}}$ and $\theta : \boldsymbol{R}_+ \to \bar{\boldsymbol{R}}$ by

$$r_M(s, t) = \sup_{\|y\| \leqq M} f^*(s, t, y)$$

$$\theta(\tau) = \sup\left\{\inf_{M>0} \frac{1}{M}\left[\|c_1\|_1 + \int_E |r_M(s, t)| d(\mu \otimes \nu)\right] \middle| E \in \mathcal{E}_S \otimes \mathcal{E}_T, (\mu \otimes \nu)(E) \leqq \tau\right\}.$$

Then θ satisfies (i)–(iv) in the above lemma.

For any $x \in F_c$ and any $y \in L^\infty(S \times T, \boldsymbol{R}^l)$, we obtain, by the Young-Fenchel inequality, that

(4) $$\begin{aligned} \int_{E_t} \langle x(s, t), y(s, t) \rangle d\mu &\leqq \int_{E_t} f(s, t, x(s, t)) d\mu + \int_{E_t} f^*(s, t, y(s, t)) d\mu \\ &\leqq c_1(t) + \int_{E_t} f^*(s, t, y(s, t)) d\mu \text{ (by (2)) for } a.e.\ t. \end{aligned}$$

Hence the following estimates hold good for any $M > 0$:

$$M\int_E \|x(s,t)\|d(\mu\otimes\nu)\leqq \sup\left\{\int_E \langle x(s,t), y(s,t)\rangle d(\mu\otimes\nu)\,\middle|\, y\in L^\infty(S\times T, \boldsymbol{R}^l), \|y\|_\infty \leqq M\right\}$$

(by Hahn-Banach theorem)

$$\leqq \sup\left\{\|c_1\|_1+\int_E f^*(s,t,y(s,t))d(\mu\otimes\nu)\,\middle|\, y\in L^\infty(S\times T, \boldsymbol{R}^l), \|y\|_\infty\leqq M\right\} \text{ (by (4))}$$

$$\leqq \|c_1\|_1+\int_E |r_M(s,t)|d(\mu\otimes\nu).$$

That is,

$$\int_E \|x(s,t)\|d(\mu\otimes\nu)\leqq \inf_{M>0}\frac{1}{M}\left\{\|c_1\|_1+\int_E|r_M(s,t)|d(\mu\otimes\nu)\right\}\leqq \theta((\mu\otimes\nu)(E)).$$

Hence, taking account of the properties of θ shown in the lemma, we can conclude that F_c is uniformly integrable. Q.E.D.

3. Existence Theorem. We shall now go over to the existence theorem for the problem $(\boldsymbol{P})$.

Assumption 1. $(S, \mathcal{E}_S, \mu)$ and $(T, \mathcal{E}_T, \nu)$ are non-atomic, complete finite measure spaces.

Assumption 2. u satisfies the following conditions.

(1) u is $(\mathcal{E}_S\otimes\mathcal{E}_T\otimes\mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\boldsymbol{R}))$-measurable.

(2) The function $x\mapsto u(s,t,x)$ is upper semi-continuous and concave for any fixed $(s,t)\in S\times T$.

(3) There exist some $a\in L^\infty(S\times T, \boldsymbol{R}^l)$ and $b\in L^1(S\times T, \boldsymbol{R})$ such that

$$u(s,t,x)\leqq\langle a(s,t), x\rangle + b(s,t)$$

for all $(s,t,x)\in S\times T\times \boldsymbol{R}^l$.

(4) $\int_{S\times T} u(s,t,x(s,t))d(\mu\otimes\nu) > -\infty$

for all $x\in L^1(S\times T, \boldsymbol{R}^l)$.

Assumption 3. $g\equiv(g^{(1)}, g^{(2)}, \cdots, g^{(k)})$ satisfies the following conditions.

(1) $g^{(i)}$ is $(\mathcal{E}_S\otimes\mathcal{E}_T\otimes\mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\bar{\boldsymbol{R}}))$-measurable.

(2) The function $x\mapsto g^{(i)}(s,t,x)$ is lower semi-continuous and convex for any fixed $(s,t)\in S\times T$.

(3) There exist some $c\in L^\infty(S\times T, \boldsymbol{R}^l)$ and $d\in L^1(S\times T, \boldsymbol{R})$ such that

$$g^{(i)}(s,t,x)\geqq\langle c(s,t), x\rangle + d(s,t).$$

for all $(s,t,x)\in S\times T\times\boldsymbol{R}^l$.

(4) $g^{(i)}$ satisfies the growth condition:

$$\mathrm{Dom}\int_{S\times T}|g^{(i)*}(s,t,y)|d(\mu\otimes\nu)=\boldsymbol{R}^l,$$

where $g^{(i)*}(s,t,\cdot)$ is the Young-Fenchel transform of $x\mapsto g^{(i)}(s,t,x)$ for each fixed $(s,t)\in S\times T$.

Assumption 4. $\omega\in L^1(T, \boldsymbol{R}^k)$.

Theorem 2. *Under Assumptions* 1–4, *our problem* $(\boldsymbol{P})$ *has an optimal solution in* $L^1(S\times T, \boldsymbol{R}^l)$.

Proof. According to Ioffe's Continuity Theorem (Ioffe [3]), Assumptions 1–2 imply that the integral functional

$$J : x \longmapsto \int_{S\times T} u(s, t, x(s, t))\, d(\mu\otimes\nu)$$

is sequentially upper semi-continuous on $L^1(S\times T, \boldsymbol{R}^l)$ with respect to the weak topology.

And Assumption 3 assures, by Theorem 1, that the set

$$F_\omega=\left\{x\in L^1(S\times T, \boldsymbol{R}^l)\,\middle|\,\int_S g(s, t, x(s, t))\,\mathrm{d}\mu\leqq\omega(t)\ \ a.e.\right\}$$

is weakly relatively compact in $L^1(S\times T, \boldsymbol{R}^l)$. Hence F_ω is L^1-bounded. Thus we obtain, by Assumption 2–(3), that

$$-\infty<\gamma\equiv\sup_{x\in F_\omega} J(x)\leqq\|a\|_\infty\cdot\sup_{x\in F_\omega}\|x\|_1+\|b\|_1\equiv C<\infty,$$

($-\infty<\gamma$ comes from Assumption 2–(4)).

Let $\{x_n\}$ be a sequence in F_ω such that

$$\lim_{n\to\infty} J(x_n)=\gamma.$$

Since F_ω is weakly relatively compact, $\{x_n\}$ has a weakly convergent subsequence. Without loss of generality, we may assume that

$$w\text{-}\lim_{n\to\infty} x_n=x^*\in L^1(S\times T, \boldsymbol{R}^l).$$

We can easily verity that $x^*\in F_\omega$ as follows. Again by the Continuity Theorem, Assumptions 1 and 3 imply that the integral functional

$$I_i : x \longmapsto \int_S g^{(i)}(s, t, x(s, t))\, d\mu$$

is sequentially lower semi-continuous on $L^1(S\times T, \boldsymbol{R}^l)$ with respect to the weak topology for any fixed $t\in T$. Hence

$$\int_S g^{(i)}(s, t, x^*(s, t))\, d\mu\leqq\liminf_n\int_S g^{(i)}(s, t, x_n(s, t))\, d\mu\leqq\omega(t),$$

from which we can conclude that $x^*\in F_\omega$.

Finally, by the sequential upper semi-continuity of J, we must have

$$J(x^*)\geqq\limsup_n J(x_n)\equiv\gamma.$$

On the other hand, it is obvious that $\gamma\geqq J(x^*)$. Hence $J(x^*)=\gamma$, which means that x^* is an optimal solution for $(\boldsymbol{P})$. Q.E.D.

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