48. Zonal Spherical Functions on the Quantum Homogeneous Space $SU_q(n+1)/SU_q(n)$

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In this note, we give an explicit expression to the zonal spherical functions on the quantum homogeneous space $SU_q(n+1)/SU_q(n)$. Details of the following arguments as well as the representation theory of the quantum group $SU_q(n+1)$ will be presented in our forthcoming paper [3]. Throughout this note, we fix a non-zero real number q.

1. Following [4], we first make a brief review on the definition of the quantum groups $SL_q(n+1; \boldsymbol{C})$ and its real form $SU_q(n+1)$.

The coordinate ring $A(SL_q(n+1; \boldsymbol{C}))$ of $SL_q(n+1; \boldsymbol{C})$ is the $\boldsymbol{C}$-algebra $A=\boldsymbol{C}[x_{ij}; 0\leq i, j\leq n]$ defined by the "canonical generators" x_{ij} $(0\leq i, j\leq n)$ and the following fundamental relations:

(1.1) $$x_{ik}x_{jk}=qx_{jk}x_{ik}, \qquad x_{ki}x_{kj}=qx_{kj}x_{ki}$$

for $0\leq i<j\leq n$, $0\leq k\leq n$,

(1.2) $$x_{il}x_{jk}=x_{jk}x_{il}, \qquad x_{ik}x_{jl}-qx_{il}x_{jk}=x_{jl}x_{ik}-q^{-1}x_{jk}x_{il}$$

for $0\leq i<j\leq n$, $0\leq k<l\leq n$ and

(1.3) $$\det_q=1.$$

The symbol $\det_q$ stands for the *quantum determinant*

(1.4) $$\det_q=\sum_{\sigma\in S_{n+1}}(-q)^{l(\sigma)}x_{0\sigma(0)}x_{1\sigma(1)}\cdots x_{n\sigma(n)},$$

where S_{n+1} is the permutation group of the set $\{0, 1, \cdots, n\}$ and, for each $\sigma\in S_{n+1}$, $l(\sigma)$ denotes the number of pairs (i, j) with $0\leq i<j\leq n$ and $\sigma(i)>\sigma(j)$. This algebra A has the structure of a Hopf algebra, endowed with the *coproduct* $\Delta: A\to A\otimes A$ and the *counit* $\varepsilon: A\to \boldsymbol{C}$ satisfying

(1.5) $$\Delta(x_{ij})=\sum_{k=0}^{n}x_{ik}\otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij})=\delta_{ij} \qquad \text{for } 0\leq i, j\leq n.$$

Moreover, there exists a unique conjugate linear anti-homomorphism $a\mapsto a^*: A\to A$ such that

(1.6) $$x_{ji}^*=S(x_{ij}) \qquad \text{for } 0\leq i, j\leq n$$

with respect to the *antipode* $S: A\to A$ of A. Together with this **-operation*, the Hopf algebra $A=A(SL_q(n+1; \boldsymbol{C}))$ defines the *-Hopf algebra $A(SU_q(n+1))$.

In what follows, we denote by G the quantum group $SU_q(n+1)$ and by K the quantum subgroup $SU_q(n)$ of $G=SU_q(n+1)$. Denote by y_{ij} $(0\leq i,$

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$j \leq n$) the canonical generators for the coordinate ring $A(K)$. Embedding of K into G is then specialized by the C-algebra epimorphism $\pi_k: A(G) \to A(K)$ such that

(1.7) $$\pi_K(x_{ij}) = y_{ij}, \quad \pi_K(x_{nn}) = 1 \quad \text{and} \quad \pi_K(x_{in}) = \pi_K(x_{nj}) = 0$$

for $0 \leq i, j < n$.

2. For a given dominant integral weight $\Lambda = \lambda_0 \varepsilon_0 + \cdots + \lambda_{n-1} \varepsilon_{n-1}$ ($\lambda_0 \geq \cdots \geq \lambda_{n-1} \geq 0$), there exists a unique irreducible right $A(G)$-comodule V_Λ with highest weight Λ. We denote by Λ_k the fundamental weight $\varepsilon_0 + \cdots + \varepsilon_{k-1}$ for $1 \leq k \leq n$. As a representation of $K = SU_q(n)$, V_Λ can be decomposed into irreducible components. It turns out that V_Λ has a trivial representation of K as an irreducible component if and only if the highest weight Λ is of the form $\Lambda = l\Lambda_1 + m\Lambda_n$ for some $l, m \in N$ and that the trivial representation may appear with multiplicity one. Such a representation V_Λ is said to be of *class 1 relative to K*.

If V_Λ is of class 1, it can be decomposed into the form

(2.1) $$V_\Lambda = C v_0 \oplus V'_\Lambda$$

as an $A(K)$-comodule, where v_0 is a K-fixed vector of V_Λ. Let $\{v_1, \cdots, v_{N-1}\}$ be a C-basis for V'_Λ ($N = \dim_C V_\Lambda$) and define the matrix elements w_{ij} of the representation V_Λ by

(2.2) $$R_G(v_j) = \sum_{i=0}^{N-1} v_i \otimes w_{ij} \qquad \text{for } 0 \leq j < N.$$

Here $R_G: V_\Lambda \to V_\Lambda \otimes A(G)$ is the structure mapping of the right $A(G)$-comodule V_Λ. Then the matrix element w_{00} does not depend on the choice of $v_0, \cdots, v_{N-1}$ and is *bi-K-invariant* in the sense that

(2.3) $$R_K(w_{00}) = w_{00} \otimes 1 \quad \text{and} \quad L_K(w_{00}) = 1 \otimes w_{00},$$

where

$$R_K = (id \otimes \pi_K) \circ \Delta \quad \text{and} \quad L_K = (\pi_K \otimes id) \circ \Delta.$$

We call w_{00} the *zonal spherical function* of V_Λ relative to K.

3. We introduce the notation of quantum r-minor determinants. Let I and J be two subsets of $\{0, 1, \cdots, n\}$ with $\sharp I = \sharp J = r$. Arrange the elements of I and J in the increasing order: $I = \{i_0, \cdots, i_{r-1}\}$ ($0 \leq i_0 < \cdots < i_{r-1} \leq n$) and $J = \{j_0, \cdots, j_{r-1}\}$ ($0 \leq j_0 < \cdots < j_{r-1} \leq n$). We define the *quantum r-minor determinant* ξ^I_J by

(3.1) $$\xi^I_J = \xi^{i_0 \cdots i_{r-1}}_{j_0 \cdots j_{r-1}} = \sum_{\sigma \in S_r} (-q)^{l(\sigma)} x_{i_0 j_{\sigma(0)}} x_{i_1 j_{\sigma(0)}} \cdots x_{i_{r-1} j_{\sigma(r-1)}}.$$

If $I = \{0, 1, \cdots r-1\}$, we use the abbreviation $\xi_J = \xi^I_J$.

To investigate spherical functions, we give a geometric realization of V_Λ (cf. [3]).

Let $\Lambda = \Lambda_{\mu_0} + \Lambda_{\mu_1} + \cdots + \Lambda_{\mu_{k-1}}$ ($\mu_0 \geq \cdots \geq \mu_{k-1} > 0$) be a dominant integral weight. Let $\boldsymbol{J} = (J_0, \cdots J_{k-1})$ be a sequence of non-empty subsets of $\{0, 1, \cdots, n\}$ with $\sharp J_s = \mu_s$ for $0 \leq s < k$. We call $\boldsymbol{J}$ a *column strict plane partition of shape* Λ if the following conditions are satisfied:

(3.2) $$\begin{cases} J_s = \{j_{0,s}, \cdots, j_{\mu_s - 1, s}\} \subset \{0, 1, \cdots, n\}, \\ j_{r,s} < j_{r+1,s} \quad \text{and} \quad j_{r,s} \leq j_{r,s+1}. \end{cases}$$

The irreducible representation of V_Λ can be realized as a right sub-$A(G)$-

comodule as

(3.3) $$V_\Lambda=\sum_J C\xi_J\subset A(G),$$

where the sum is taken over the set of all column strict plane partitions of shape Λ and

(3.4) $$\xi_J=\xi_{J_0}\cdots\xi_{J_{k-1}} \qquad \text{if } J=(J_0,\cdots,J_{k-1}).$$

It is seen that these vectors ξ_J are linearly independent and that they form a C-basis for a right $A(G)$-comodule. Note that the product of minor determinants

(3.5) $$\xi_{01\cdots\mu_0-1}\cdots\xi_{01\cdots\mu_{k-1}-1}$$

gives the highest weight vector of V_Λ. We remark that V_Λ is identified with the vector space of all left relative B_--invariants in $A(SL_q(n+1;C))$ with respect to the character corresponding to Λ, where B_- is the Borel subgroup "of lower triangular matrices" (see [3]).

If $\Lambda=l\Lambda_1+m\Lambda_n$ $(l,m\in N)$, the spherical representation V_Λ contains a K-fixed vector

(3.6) $$v_0=(\xi^0_n)^l(\xi^{01\cdots n-1}_{01\cdots n-1})^m.$$

By using algebraic properties of quantum minor determinants, we can determine explicitly the zonal spherical function of V_Λ in terms of *basic hypergeometric series* ${}_2\varphi_1$:

(3.7) $${}_2\varphi_1\begin{pmatrix}a,b\\c\end{pmatrix};q,z\Big)=\sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k}z^k,\qquad (a;q)_k=\prod_{j=0}^{k-1}(1-aq^j).$$

Theorem. *Let V_Λ be the representation of $G=SU_q(n+1)$ with highest weight $\Lambda=l\Lambda_1+m\Lambda_n$ $(l,m\in N)$. Then the zonal spherical function $w=w_{00}$ of V_Λ relative to $K=SU_q(n)$ is expressed by a basic hypergeometric series in $z=1-x_{nn}\xi^{01\cdots n-1}_{01\cdots n-1}$ as follows:*

(3.8) $$w=(x_{nn})^{l-m}{}_2\varphi_1\left(\begin{matrix}q^{-2m},\ q^{2(l+n)}\\ q^{2n}\end{matrix};q^2,q^2z\right)\qquad \text{if } l\geq m,$$

and

(3.9) $$w={}_2\varphi_1\left(\begin{matrix}q^{-2l},\ q^{2(m+n)}\\ q^{2n}\end{matrix};q^2,q^2z\right)(\xi^{01\cdots n-1}_{01\cdots n-1})^{m-l}\qquad \text{if } l\leq m.$$

The above polynomials in z are so-called *little q-Jacobi polynomials*. As for *zonal* spherical functions, Theorem generalizes a result of [1, 2].

References

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