47. A Note on Irreducible Representations of Profinite Nilpotent Groups

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1. The purpose of this work is to parametrize the set of isomorphism classes of complex continuous finite dimensional irreducible representations of a profinite nilpotent group G by certain characters of the Lie ring $L(G)$ of G which is formed from the lower central series of G. Since every component $L_i(G)$ of $L(G)$ is a certain quotient of $T_i(G^{ab})$, the i-fold tensor product of $G^{ab}=G/[G,G]$, this implies that the irreducible representations of G are determined by certain characters of G^{ab}.

2. Let G be a profinite nilpotent group, and for every integer $c\geq 1$, denote by $I^c(G)$ the set of isomorphism classes of (complex continuous finite dimensional) irreducible representations of G such that their finite images are nilpotent of class c. Put

$$I(G):=\bigcup_{c\geq 1} I^c(G).$$

Denote the closed commutator subgroup of G by $[G,G]$ and put

$$G^{ab}=G/[G,G],$$

$$T_i(G^{ab})=i\text{-fold tensor product of } G^{ab},$$

$$T^c(G)=\prod_{i=1}^{c} T_i(G^{ab}), \qquad T(G)=\prod_{i\geq 1} T_i(G^{ab}).$$

For a locally compact abelian group A denote its Pontrjagin dual by $A^\wedge$. We shall show the substantial contents of the following statement in the sequel of the proof:

Theorem 1. *There are quotients $\bar{T}^c(G)$ and $\bar{T}(G)$ of $T^c(G)$ and $T(G)$, respectively, which are determined by certain relations between commutators of G, and surjective maps*

$$\bar{T}^c(G)^\wedge \longrightarrow \bigcup_{i=1}^{c} I^c(G), \qquad \bar{T}(G)^\wedge \longrightarrow I(G).$$

Remark. A preliminary version of this result is contained in [3], § 9, and showed on the basis of Clifford's theory (e.g. [1]-V, or [3], § 5, for the profinite case) and the results of Yamazaki [4] on projective representations of finite groups. However, we give here a different proof based on the results of Iwahori and Matsumoto [2] which shows that the maps may be considered canonically.

3. In the proof of the theorem we use the following notation. Let

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$$G=G_1 \supseteq G_2=[G_1, G] \supseteq G_3=[G_2, G] \supseteq \cdots$$

be the lower central series of G and put

$$L_i(G)=G_i/G_{i+1},\ L^c(G)=\prod_{i=1}^{c} L_i(G),\ L(G)=\prod_{i\geq 1} L_i(G).$$

It follows from the definition that

$$x_1 \otimes x_2 \otimes \cdots \otimes x_i \longrightarrow [\cdots[[x_1, x_2], x_3], \cdots, x_i]$$

induces an epimorphism

$$T_i(G^{ab}) \longrightarrow L_i(G).$$

The kernel consists of relations between the commutators of G in G_i modulo G_{i+1}. Therefore, the theorem is an immediate consequence of the following proposition the proof of which is given in Section 5.

Proposition 1. (i) *$I^1(G)$ is canonically identified with $L^1(G)^\wedge$.* (ii) *For each $c\geq 2$, the members of $I^c(G)$ are fully parametrized by the elements of $L^c(G)^\wedge - L^{c-1}(G)^\wedge$.*

4. The incidence correspondence. In this section, we consider a profinite group H and its closed normal subgroup N such that the quotient group $A=H/N$ is abelian, and establish one of the basic results of Iwahori and Matsumoto [2], Theorem 4.13, also in the case where A is infinite. Let S and T be complex continuous finite dimensional irreducible representations of H and N, respectively; after Iwahori and Matsumoto we say that S and T are incident if T is equivalent to an irreducible component of the restriction, $\mathrm{res}(S)$, of S to N; denote the multiplicity of T in $\mathrm{res}(S)$ by $(\mathrm{res}(S):T)$; then S and T are incident if and only if $(\mathrm{res}(S):T)\geq 1$. We denote the equivalence classes of S and T by $[S]$ and $[T]$, respectively.

Now the set $I(H)$ of all equivalence classes of complex continuous finite dimensional irreducible representations of H is acted by $A^\wedge$ by twisting-multiplication, on one hand. The isotropy subgroup of an element $[S]$ is denoted by $A^\wedge{}_{[S]}$; we will soon see that this is always a finite group. On the other hand, the quotient group A itself acts on $I(N)$ as follows: for $g\in H$ and $[T]\in I(N)$ define $[T]^g$ to be the class of $T^g(x):=T(g^{-1}xg)$, $x\in N$; obviously this induces the action of A on $I(N)$. The isotropy subgroup of an element $[T]$ is denoted by $A_{[T]}$.

Theorem 2. *Let H be a profinite group, and N be a closed normal subgroup such that $A=H/N$ is abelian. Then,* (I) *for every complex finite dimensional irreducible representation S of H, there is an irreducible representation T of N such that* $(\mathrm{res}(S):T)\geq 1$; *$T$ is unique up to the action of A; the isotropy group $A_{[T]}$ is a closed subgroup of A of finite index. Conversely,* (II) *for every complex finite dimensional irreducible representation T of N, there exists an irreducible representation S of H which is incident to T; S is unique up to the action of $A^\wedge$; the isotropy group $A^\wedge{}_{[S]}$ is finite. Thus, there exists a canonical bijection between the two sets of orbits $I(N)/A$ and $I(H)/A^\wedge$. Moreover,* (III) *if S and T are as above incident, then the annihilator $A^\perp_{[T]}$ of $A_{[T]}$ lies in $A^\wedge{}_{[S]}$, and* *the index is determined by*

$$[A^\frown{}_{[S]} : A^{\perp}_{[T]}] = (\mathrm{res}(S) : T)^2.$$

Proof. Let H, N and A be as in the theorem. We can reduce Theorem 2 to the case where H is finite, that is, to Theorem 4.13 of Iwahori and Matsumoto [2], by the usual way. First let S and S' be those irreducible representations of H both of which are incident to the same irreducible representation of N. Then the quotient group $H/\mathrm{Ker}(S)\cap \mathrm{Ker}(S')$ is finite. Hence by Theorem 4.13 of [2] we conclude that S' is a multiple of S by an inflated element of $A^\frown$ from the dual group of $A/\{(\mathrm{Ker}(S)\cap \mathrm{Ker}(S'))\cdot N/N\}$. Next suppose that an irreducible representation T of N is given. Then $\mathrm{Ker}(T)$ is an open subgroup of N. Therefore there is an open normal subgroup U of H such that $U\cap N\subset \mathrm{Ker}(T)$. Put

$$\bar{H} := H/U, \ \bar{N} := NU/U \quad \text{and} \quad \bar{A} := \bar{H}/\bar{N}.$$

Then we have an irreducible representation $\bar{T}$ of $\bar{N}$ determined by T because $\bar{N}\cong N/U\cap N$. Take an irreducible component $\bar{S}$ of the induced representation of $\bar{H}$ from $\bar{T}$. Then by the Frobenius reciprocity law, we see that $\bar{S}$ and $\bar{T}$ are incident. Let S be the inflation of $\bar{S}$ on H. It is obvious that S is a continuous irreducible representation of H which is incident to T. Now for $g\in U$ and $x\in N$, we have

$$T^g(x) = T(g^{-1}xg) = T(x\cdot x^{-1}g^{-1}xg) = T(x)$$

because $x^{-1}g^{-1}xg$ belongs to the subgroup $U\cap N$ of $\mathrm{Ker}(T)$. Hence the isotropy group $A_{[T]}$ contains NU/N; furthermore, it is clear that $A_{[T]}/NU$ is none other than $\bar{A}_{[\bar{T}]}$. Next let φ be the inflation of an element of $A^\frown$ to H. Then it is obvious that $\varphi\otimes S$ is equivalent to S only if $\mathrm{Ker}(\varphi)$ contains $\mathrm{Ker}(S)$ in which U lies. Therefore φ has to be the inflation of an element of $\bar{A}^\frown$. This shows that the isotropy group $A^\frown{}_{[S]}$ is the image of the finite group $\bar{A}^\frown{}_{[\bar{S}]}$ by the inflation map. Our theorem is now completely reduced to the finite case of $\bar{H}$, $\bar{N}$ and $\bar{A}$, and easily verified by Theorem 4.13 of [2].

5. Proof of Proposition 1. (i) is obvious. To show (ii), first we fix a "*section*" $\gamma_{i+1}: I(G_{i+1})\to I(G_i)$ for each $i=1,2,3,\cdots$, as follows: by Theorem 2, there exists a canonical surjection from $I(G_{i+1})$ onto the set of orbits $I(G_i)/L_i(G)^\frown$; choose a section of the natural projection of $I(G_i)$ onto the set of orbits which assign its representative to each orbit; then the composition of these two maps is our *section* γ_{i+1}. It is clear that

(5.1) γ_{i+1} maps $I^c(G_{i+1})$ into $I^{c+1}(G_i)$ for each $c\geq 1$; for each $[T]\in I^c(G_{i+1})$, the orbit $L_i(G)^\frown\cdot\gamma_{i+1}([T])$ consists of all those classes the members of which are incident to T.

Now we define a map $\theta_c : L^c(G)^\frown - L^{c-1}(G)^\frown\to I^c(G)$ for $c\geq 2$. Let φ be an element of form

$$\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_c)\in L^c(G)^\frown, \quad \varphi_i\in L_i(G)^\frown, \quad \varphi_c\neq 1.$$

Let $D_c=\inf(\varphi_c)$ be the linear representation of G_c naturally determined by φ_c. Then $\gamma_c([D_c])$ belongs to $I^2(G_{c-1})$. Therefore φ_{c-1} determines a class of representation $[D_{c-1}]=\varphi_{c-1}\cdot\gamma_c([D_c])\in I^2(G_{c-1})$. In this manner, successively,

φ finally determines an equivalent class $[D_1]$ of irreducible representation in $I^c(G)$. We assign $\theta_c(\varphi)=[D_1]$ to φ to define the map θ_c. Conversely, suppose that $[D_1]\in I^c(G)$ is given. Take $[D_2]\in I^{c-1}(G_2)$ such that D_1 and D_2 are incident. Then there is an element $\varphi_1\in L_1(G)^\wedge$ to satisfy $[D_1]=\varphi_1\cdot\gamma_2([D_2])$. Next we can find those D_3 and φ_2 for D_2 which satisfy $[D_2]=\varphi_2\cdot\gamma_3([D_3])$, and so on, and finally obtain a series of elements $\varphi_1, \varphi_2, \cdots, \varphi_{c-1}$ and $[D_c]\in I^1(G_c)$. Since D_c is a linear character, it certainly gives $\varphi_c\in L_c(G)^\wedge$; this cannot be trivial because $D_1(G)$ is nilpotent of class c by assumption. Thus we have found an element

$$\varphi=(\varphi_1, \varphi_2, \cdots, \varphi_c)\in L^c(G)^\wedge, \quad \varphi_i\in L_i(G)^\wedge, \quad \varphi_c\neq 1,$$

which is sent to $[D_1]$ by the above constructed map θ_c. This shows that θ_c is surjective. Proposition 1 is now proved.

Remark 1. The proposition is easily modified for a profinite solvable group if its derived series is taken in place of the lower central series.

Remark 2. If we apply our theorem to the case where G is the Galois group of the maximal nilpotent extension of a number field k, we see from class field theory that the elements D in $I(G)$ are determined by certain characters of the idele class group of k, and it is an important task to determine the ramification properties of D from those of the corresponding idele class group characters.

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