# 40. A Spectral Decomposition of the Product of Four Zeta-values

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The aim of this note is to inject Kuznecov's trace formula [2] into the argument of our former work [4], and to make a preparation which will be needed in our plan of a finer study of the fourth power moment of the Riemann zeta-function over the critical line.

We consider the product $\zeta(u)\zeta(v)\zeta(w)\zeta(z)$. In the region of absolute convergence this is decomposed into three parts:

$$\{\sum_{km=ln}+\sum_{km<ln}+\sum_{km>ln}\}k^{-u}l^{-v}m^{-w}n^{-z}.$$

The first sum can be computed by means of Ramanujan's identity. Let us denote the second sum by $g(u,v,w,z)$; then the third is $g(v,u,z,w)$. We put

$$\begin{aligned} g^*(u,v,w,z) &= g(u,v,w,z)-\Gamma(z)^{-1}\Gamma(1-w)\Gamma(w+z-1)\zeta(u+v)\\ &\quad\times\zeta(w+z-1)\zeta(u-w+1)\zeta(v-z+1)\{\zeta(u+v-w-z+2)\}^{-1}\\ &\quad-2(2\pi)^{w-u}\cos\left(\frac{\pi}{2}(u-w)\right)\Gamma(z)^{-1}\Gamma(u-w)\Gamma(1-u)\Gamma(u+z-1)\\ &\quad\times\zeta(u+z-1)\zeta(v+w)\zeta(u-w)\zeta(v-z+1)\{\zeta(v+w-u-z+2)\}^{-1}. \end{aligned}$$

Then we are going to show that an analytic continuation of $g^*$ can be described in terms of sums of products of Hecke $L$-series.

To state our result we have to introduce some terminologies from the theory of automorphic functions: Let $\{\chi_j^2+(1/4);\chi_j>0\}\cup\{0\}$ be the discrete spectrum of the non-Euclidean Laplacian acting on the usual Hilbert space of $L^2$ automorphic functions with respect to the full modular group. Let $\varphi_j$ be the Maass wave form attached to $\chi_j$. With the first Fourier coefficient $\rho_j(1)$ of $\varphi_j$ we put $\alpha_j=|\rho_j(1)|^2(\cos(i\pi\chi_j))^{-1}$. Also, $H_j$ is the Hecke $L$-series corresponding to $\varphi_j$, and $\varepsilon_j$ is the parity sign of $\varphi_j$. Next, let $\{\varphi_{j,2k};1\leqq j\leqq d_{2k}\}$ be the orthonormal base, consisting of eigen functions of Hecke operators $T_{2k}(n)$, of the usual unitary space of regular cusp forms of weight $2k$ with respect to the full modular group. With the first Fourier coefficient $\rho_{j,2k}(1)$ of $\varphi_{j,2k}$ we put $\alpha_{j,2k}=(4\pi)^{1-2k}(2k-1)!|\rho_{j,2k}(1)|^2$. Finally, $H_{j,2k}$ is the Hecke $L$-series corresponding to $\varphi_{j,2k}$.

Further, let $\theta>1$ be a parameter, and let $A_\theta$ be the domain

$$\left\{(u,v,w,z);\,2\mathrm{Re}(z)>\mathrm{Re}(u+v+w+z)>\frac{3}{2}+2\theta,\ \mathrm{Re}(u+z)<\theta,\ \mathrm{Re}(w+z)<\theta\right\}.$$

In $A_\theta$ we define two functions $\Psi_\theta$ and $\Phi_\theta$ by

$$\Psi_\theta(u,v,w,z;\xi)=-i(2\pi)^{2+z-v}\Gamma(z)^{-1}\cos\left(\frac{\pi}{2}(u-w)\right)$$
$$\times\int_{\theta-i\infty}^{\theta+i\infty}\Gamma(r)\Gamma(z-r)\Gamma(1-u-z+r)\Gamma(1-w-z+r)$$
$$\times\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r+\xi\right)\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r-\xi\right)$$
$$\times\sin\left(\pi\left(\frac{1}{2}(u+v+w+z)-r\right)\right)dr,$$
$$\Phi_\theta(u,v,w,z;\xi)=-i(2\pi)^{2+z-v}\Gamma(z)^{-1}\cos(\pi\xi)$$
$$\times\int_{\theta-i\infty}^{\theta+i\infty}\Gamma(r)\Gamma(z-r)\Gamma(1-u-z+r)\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r+\xi\right)$$
$$\times\Gamma(1-w-z+r)\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r-\xi\right)$$
$$\times\cos\left(\pi\left(z+\frac{1}{2}(u+w)-r\right)\right)dr.$$

We note that for $(u,v,w,z)\in A_\theta$ and $|\mathrm{Re}\,\xi|<5/4$ both $\Psi_\theta$ and $\Phi_\theta$ are regular and decay as $O(|\xi|^{-2+\mathrm{Re}(u+v+w-z)})$.

Then we have the following:

**Theorem 1.** *For any $\theta>1$ $g^*$ can be continued analytically to the domain $A_\theta$, and there we have*

$$g^*(u,v,w,z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}(u+v+w+z-1)+i\xi\right)$$
$$\times\zeta\left(\frac{1}{2}(u+v+w+z-1)-i\xi\right)\zeta\left(\frac{1}{2}(u+v-w-z+1)+i\xi\right)$$
$$\times\zeta\left(\frac{1}{2}(u+v-w-z+1)-i\xi\right)\zeta\left(\frac{1}{2}(v+w-u-z+1)+i\xi\right)$$
$$\times\zeta\left(\frac{1}{2}(v+w-u-z+1)-i\xi\right)|\zeta(1+2i\xi)|^{-2}(\Psi_\theta-\Phi_\theta)(u,v,w,z;i\xi)\,d\xi$$
$$+\sum_{j=1}^{\infty}\alpha_j H_j\left(\frac{1}{2}(u+v+w+z-1)\right)H_j\left(\frac{1}{2}(u+v-w-z+1)\right)$$
$$\times H_j\left(\frac{1}{2}(v+w-u-z+1)\right)(\Psi_\theta-\varepsilon_j\Phi_\theta)(u,v,w,z;i\chi_j)$$
$$+\sum_{k=6}^{\infty}\sum_{j=1}^{d_{2k}}\alpha_{j,2k}H_{j,2k}\left(\frac{1}{2}(u+v+w+z-1)\right)H_{j,2k}\left(\frac{1}{2}(u+v-w-z+1)\right)$$
$$\times H_{j,2k}\left(\frac{1}{2}(v+w-u-z+1)\right)\Psi_\theta\left(u,v,w,z;k-\frac{1}{2}\right).$$

The domain $A_\theta$ is chosen to ensure amply the absolute convergence everywhere in the above. But, then arises the difficult problem of finding an analytic continuation of the right side to some domain containing well the most interesting case $\{u=1-z, v=z, w=1-z, \mathrm{Re}(z)=1/2\}$, which is vital if we try to extend Atkinson's result [1, Theorem] to the fourth power mean situation. Though we have not been able to solve this, we can show that a mean value of $g^*$ does admit such a continuation. For this sake it should be observed that if we replace $(u, v, w, z)$ by $(u+a, v-a, w+a, z-a)$ in the above expression for $g^*$ all factors except for some $\Gamma$-factors in $\Psi_\theta$ and $\Phi_\theta$ remain unchanged. Being this noted, we put, for arbitrary positive $T$ and $\Delta$,

$$G^*(T, \Delta; u, v, w, z)=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty} \exp\left(-\left(\frac{T-t}{\Delta}\right)^2\right)$$
$$\times g^*(u+it, v-it, w+it, z-it)\,dt.$$

Accordingly we take the means $\tilde{\Psi}_\theta$ and $\tilde{\Phi}_\theta$ of $\Psi_\theta$ and $\Phi_\theta$: We have

$$\tilde{\Psi}_\theta(T, \Delta; u, v, w, z; \xi)=-i(2\pi)^{2+z-v}\cos\left(\frac{\pi}{2}(u-w)\right)$$
$$\times\int_0^\infty (1+x)^{-z+iT}\exp\left(-\left(\frac{\Delta}{2}\log(1+x)\right)^2\right)$$
$$\times\int_{\theta-i\infty}^{\theta+i\infty} x^{r-1}\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r+\xi\right)\Gamma\left(\frac{1}{2}(u+v+x+z-1)-r-\xi\right)$$
$$\times\Gamma(1-w-z+r)\Gamma(1-u-z+r)\sin\left(\pi\left(\frac{1}{2}(u+v+w+z)-r\right)\right)drdx,$$

$$\tilde{\Phi}_\theta(T, \Delta; u, v, w, z; \xi)=-i(2\pi)^{2+z-v}\cos(\pi\xi)\int_0^\infty (1+x)^{-z+iT}$$
$$\times\exp\left(-\left(\frac{\Delta}{2}\log(1+x)\right)^2\right)\int_{\theta-i\infty}^{\theta+i\infty} x^{r-1}\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r+\xi\right)$$
$$\times\Gamma\left(\frac{1}{2}(u+v+w+z-1)-r-\xi\right)\Gamma(1-w-z+r)\Gamma(1-u-z+r)$$
$$\times\cos\left(\pi\left(z+\frac{1}{2}(u+w)-r\right)\right)drdx.$$

These are regular, for any $\theta>1$ and $|\mathrm{Re}\,\xi|<5/4$, in the new domain $B_\theta$:

$$\left\{(u, v, w, z);\ \mathrm{Re}\,(u+v+w+z)>\frac{3}{2}+2\theta,\ \mathrm{Re}\,(u+z)<\theta,\ \mathrm{Re}\,(w+z)<\theta\right\},$$

where they decay as $O(|\xi|^{-c})$ for any fixed $c>0$ when $|\xi|$ tends to infinity. We may compute these $r$-integrals in terms of hypergeometric functions, but have left them in crude form for the sake of easy checking of absolute convergence and order estimation.

Then we have

**Theorem 2.** *For any $\theta>1$ $G^*$ can be continued analytically to the domain $B_\theta$, and there we have the expression for $G^*$ which is the result of replacing $\Psi_\theta$ and $\Phi_\theta$ by $\tilde{\Psi}_\theta$ and $\tilde{\Phi}_\theta$, respectively, on the right side of the formula for $g^*$ in Theorem* 1.

Though the domain $B_\theta$, $\theta>1$, does not contain the most important point $(1/2, 1/2, 1/2, 1/2)$, we need, in fact, only a minor modification in our expression for $G^*$ in order to attain a continuation which allows this specialization. This will be given in our next note, and the details will be published elsewhere.

Another possible application of Theorem 1 is to the sum

$$\sum_{k,l,m,n} f(k, l, m, n),$$

where $f$ is an arbitrary $C^\infty$ function with a compact support in $(0, \infty)^4$. Taking the Mellin transform of $f$ we immediately see that this is closely related to our topics. Theorem 1 yields a bizarre transformation formula for the sum, which will be discussed somewhere else.

It should be stressed that the topics of Kuznecov [3] and Zavorotni [5] come close to ours. However, our argument starts from Atkinson's dissection device, and differs considerably from theirs. Also we should remark that our argument can easily be extended so as to include Dirichlet $L$-functions into discussion, though there are some problems pertaining to congruence subgroups.

## References

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