39. Zeta Zeros, Hurwitz Zeta Functions and $L(1,\chi)$

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§1. **Introduction.** Let a be a positive number <1. We are concerned with the value distribution of the Hurwitz zeta function $\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$ (for $\mathrm{Re}(s)>1$), at the zeros of the Riemann zeta function $\zeta(s)$.

Although $\zeta(s,a)$ has many good properties like $\zeta(s)$, it fails to have the Euler product formula except when $a=1/2$, in which case we have $\zeta(s,1/2)=(2^s-1)\zeta(s)$. So it might be interesting to clarify how any problem concerning $\zeta(s,a)$ depends on a. We assume the Riemann Hypothesis throughout this article and prove the following theorem. To state our theorem, we put $L_a(1)=\sum_{n=1}^{\infty}\frac{e(-na)}{n}$ with $e(y)=e^{2\pi iy}$ and $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geq 1$, and $=0$ otherwise. We denote the imaginary parts of the zeros of $\zeta(s)$ by γ.

Theorem. *For any positive* $a<1$,

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\leq T}\zeta\left(\frac{1}{2}+i\gamma,a\right)=-\Lambda\left(\frac{1}{a}\right)-L_a(1).$$

From this theorem we see first that for any integer $k\geq 2$,

$$\begin{aligned}&1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\frac{k-1}{k}+\frac{1}{k+1}+\frac{1}{k+2}+\cdots\\&+\frac{1}{2k-1}-\frac{k-1}{2k}+\frac{1}{2k+1}+\cdots\\&=\log k,\end{aligned}$$

since $\sum_{b=1}^{k-1}\zeta(s,b/k)=(k^s-1)\zeta(s)$ and $\sum_{b=1}^{k-1}\Lambda(k/b)=\sum_{m|k}\Lambda(m)=\log k$. (We know, of course, that this can be proved in an elementary way.)

We see next that for any primitive character $\chi \bmod q\geq 3$,

$$\begin{aligned}&\lim_{T\to\infty}\frac{2\pi}{T}\sum_{b=1}^{q-1}\bar{\chi}(b)\sum_{0<\gamma\leq T}\zeta\left(\frac{1}{2}+i\gamma,\frac{b}{q}\right)\\&=-\sum_{b=1}^{q-1}\bar{\chi}(b)\Lambda\left(\frac{q}{b}\right)-\sum_{b=1}^{q-1}L_{b/q}(1)\bar{\chi}(b)\\&=-\Lambda(q)-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{b=1}^{q-1}e\left(-\frac{b}{q}n\right)\bar{\chi}(b)\\&=-\Lambda(q)-\bar{\tau}(\chi)L(1,\chi),\end{aligned}$$

where $L(s,\chi)$ is the Dirichlet L-function and $\tau(\chi)=\sum_{b=1}^{q}\chi(b)e(b/q)$. Moreover since $\zeta(s,b/q)$ can be written as a linear combination of L-functions, we get the following new expressions of $L(1,\chi)$ (cf. also [5] and [6] for other type of expressions).

Corollary. *For any primitive character* $\chi \bmod q\geq 3$,

$$L(1,\chi)=-\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\le T}\left(\sum_{b=1}^{q-1}\bar{\chi}(b)\zeta\left(\frac{1}{2}+i\gamma,\frac{b}{q}\right)-q^{(1/2)+i\gamma}\right)\frac{1}{\bar{\tau}(\chi)}$$
$$=-\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\le T}\frac{q^{(1/2)+i\gamma}}{\bar{\tau}(\chi)}\left(L\left(\frac{1}{2}+i\gamma,\bar{\chi}\right)-1\right).$$

We should remark that this corollary can be proved directly by evaluating the sum $\sum_{0<\gamma\le T}x^{(1/2)+i\gamma}(L((1/2)+i\gamma,\bar{\chi})-1)$. In fact, it should be compared with the result (cf. [2] and [3]) for $x=1$;

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\le T}\left(L\left(\frac{1}{2}+i\gamma,\bar{\chi}\right)-1\right)=-L(1,\chi)\bar{\tau}(\chi)\frac{\mu(q)}{\varphi(q)}+\frac{L'}{L}(1,\bar{\chi}),$$

where $\mu(q)$ is the Möbius function and $\varphi(q)$ is the Euler function. We remark also that our results can be extended to the zeros of Dirichlet L-functions. These will appear elsewhere.

To prove our theorem we shall use the following lemmas which are the refinements of Theorems 1′ and 2′ in [2] and can be obtained by refining the author's proof in [1] (cf. [4]).

Lemma A. *For $x>1$ and $T>T_0$, we have*

$$\sum_{0<\gamma\le T}x^{i\gamma}=-\frac{T}{2\pi}\frac{\Lambda(x)}{\sqrt{x}}+M(x,T)+O(\sqrt{x}\log(2x))$$
$$+O\left(\frac{1}{\sqrt{x}}\sum_{\substack{(x/2)<n<2x\\ n\ne x}}\Lambda(n)\,\mathrm{Min}\left(T,\frac{1}{\left|\log\left(\frac{x}{n}\right)\right|}\right)\right)$$
$$+O\left(\sqrt{x}\sqrt{\frac{\log T}{\log\log T}}\right)+O\left(x^{1/\log\log T}\log(2x)\frac{\log T}{\log\log T}\right),$$

where

$$M(x,T)\equiv\frac{1}{2\pi}\int_1^T x^{it}\log\left(\frac{t}{2\pi}\right)dt$$
$$=\begin{cases}\dfrac{x^{iT}\log(T/2\pi)}{2\pi i\log x}+O\left(\dfrac{1}{\log x}+\dfrac{1}{\log^2 x}\right) & \text{if } \dfrac{1}{\log T}\ll\log x\\ O\left(\mathrm{Min}\left(\dfrac{\log T}{\log x},T\log T\right)\right) & \text{if } \log x\ll\dfrac{1}{\log T}.\end{cases}$$

Lemma B. *Suppose that $0<(2\pi\alpha/b)\le Y<T$, $(T/2\pi\alpha)\gg 1$ and $T>T_0$. Then we have for any positive $b\le 2$,*

$$\sum_{Y<\gamma\le T}e\left(\frac{b\gamma}{2\pi}\log\frac{b\gamma}{2\pi e\alpha}\right)$$
$$=-e^{\pi i/4}\frac{\sqrt{\alpha}}{b}\sum_{(Yb/2\pi\alpha)^b\le n\le(Tb/2\pi\alpha)^b}\Lambda(n)n^{(1/2)((1/b)-1)}e(-\alpha n^{1/b})$$
$$+O\left(T^{2/5}\left(\frac{T}{2\pi\alpha}\right)^{b/2}\right)+O\left(\left(T^{1/2}\left(\frac{T}{2\pi\alpha}\right)^{-b/2}+Y^{1/2}\left(\frac{Y}{2\pi\alpha}\right)^{-b/2}\right)\log\frac{T}{2\pi\alpha}\right)$$
$$+O\left(\log T\cdot\mathrm{Min}\left(\frac{1}{\log\frac{Y}{2\pi\alpha}},\sqrt{\alpha}+1\right)\right).$$

§2. Proof of Theorem. We suppose that $(1/T)\ll\log 1/\alpha$. Using the approximate functional equation of $\zeta(s,\alpha)$ due to Rane (cf. p. 204 of [9]), we get

$$\sum_{0<\gamma\le T}\zeta\left(\frac{1}{2}+i\gamma, a\right)=\sum_{0<\gamma\le T}\sum_{0\le n\le\sqrt{\gamma/2\pi}}(n+a)^{-(1/2)-i\gamma}$$
$$+\sum_{0<\gamma\le T}e\left(-\frac{\gamma}{2\pi}\log\frac{\gamma}{2\pi e}\right)e^{\pi i/4}\sum_{1\le m\le\sqrt{\gamma/2\pi}}e(-ma)m^{-(1/2)+i\gamma}$$
$$+O(\sum_{0<\gamma\le T}\gamma^{-1/4})$$
$$=a^{-1/2}\sum_{0<\gamma\le T}a^{-i\gamma}+\sum_{1\le n\le\sqrt{T/2\pi}}(n+a)^{-1/2}\sum_{2\pi n^2\le\gamma\le T}(n+a)^{-i\gamma}$$
$$+e^{\pi i/4}\sum_{1\le m\le\sqrt{T/2\pi}}e(-ma)m^{-1/2}\sum_{2\pi m^2\le\gamma\le T}e\left(-\frac{\gamma}{2\pi}\log\frac{\gamma}{2\pi em}\right)$$
$$+O(T^{3/4}\log T)$$
$$=S_1+S_2+S_3+O(T^{3/4}\log T),\ \text{say.}$$

Using Lemma A, we get

$$S_1=-\frac{T}{2\pi}\Lambda\left(\frac{1}{a}\right)+O\left(\frac{1}{a}\log\left(\frac{2}{a}\right)\right)+O\left(\frac{\log T}{\log\frac{1}{a}\cdot\sqrt{a}}\right)$$
$$+O\left(\sum_{\substack{(1/2a)<n<2(1/a)\\ n\ne(1/a)}}\Lambda(n)\operatorname{Min}\left(T,\frac{1}{\left|\log\left(\frac{1}{na}\right)\right|}\right)\right)$$
$$+O\left(\frac{1}{a}\sqrt{\frac{\log T}{\log\log T}}\right)+O\left(\frac{1}{\sqrt{a}}\left(\frac{1}{a}\right)^{1/\log\log T}\log\left(\frac{2}{a}\right)\frac{\log T}{\log\log T}\right)$$
$$=-\frac{T}{2\pi}\Lambda\left(\frac{1}{a}\right)+O\left(\frac{1}{a}\log\left(\frac{2}{a}\right)\log\log\left(\frac{3}{a}\right)\right)+O\left(\frac{\log T}{\log\frac{1}{a}\cdot\sqrt{a}}\right)$$
$$+O\left(\frac{1}{a}\sqrt{\frac{\log T}{\log\log T}}\right)+O\left(\Lambda(n(a))\operatorname{Min}\left(T,\frac{1/a}{\left|\frac{1}{a}-n(a)\right|}\right)\right)$$
$$+O\left(\frac{1}{\sqrt{a}}\left(\frac{1}{a}\right)^{1/\log\log T}\log\left(\frac{2}{a}\right)\frac{\log T}{\log\log T}\right),$$

where $n(a)$ is the nearest integer to $1/a$ other than $1/a$ itself.

Using Lemma A, we get also

$$S_2=\sum_{1\le n\le\sqrt{T/2\pi}}\frac{1}{\sqrt{n+a}}\left\{O(\sqrt{n+a}\log(n+a))+O\left(\frac{\log T}{\log(n+a)}\right)\right.$$
$$+O\left(\frac{1}{\sqrt{n+a}}\sum_{\substack{(1/2)(n+a)<m<2(n+a)\\ m\ne n+a}}\Lambda(m)\frac{1}{\left|\log\left(\frac{m}{n+a}\right)\right|}\right)$$
$$\left.+O\left((n+a)^{1/\log\log T}\log(2(n+a))\frac{\log T}{\log\log T}\right)\right\}$$
$$=O(\sqrt{T}\log T)+O\left(\sum_{1\le n\le\sqrt{T/2\pi}}\sum_{\substack{(1/2)(n+a)<m<2(n+a)\\ m\ne n,n+1}}\frac{\Lambda(m)}{|n+a-m|}\right)+O\left(\sum_{1\le n\le\sqrt{T/2\pi}}\frac{\Lambda(n)}{\|a\|}\right)$$
$$=O\left(\sqrt{T}\left(\log T\cdot\log\log T+\frac{1}{\|a\|}\right)\right),$$

where $\|a\|=\operatorname{Min}(a,1-a)$.

Using Lemma B, we get

$$S_3 = e^{\pi i/4} \sum_{1\le m\le \sqrt{T/2\pi}} e(-ma)m^{-1/2}\Big\{-e^{-\pi i/4}\sqrt{m} \sum_{m<n<(T/2\pi m)} \Lambda(n)$$

$$+O\Big(T^{9/10}\frac{1}{\sqrt{m}}\Big)+O(\sqrt{m}\log T)+O\Big(\frac{\log T}{\log(3m)}\Big)\Big\}$$

$$=-\frac{T}{2\pi}\sum_{1\le m\le \sqrt{T/2\pi}}\frac{e(-ma)}{m}+\sum_{1\le m\le \sqrt{T/2\pi}} m\cdot e(-ma)+O(T^{9/10}\log T)$$

$$=-\frac{T}{2\pi}\sum_{m=1}^{\infty}\frac{e(-ma)}{m}+O\Big(\frac{\sqrt{T}}{\|a\|}\Big)+O(T^{9/10}\log T).$$

Thus if we fix a, then we get

$$\sum_{0<\gamma\le T}\zeta\Big(\frac{1}{2}+i\gamma, a\Big)=-\frac{T}{2\pi}\Big(\Lambda\Big(\frac{1}{a}\Big)+L_a(1)\Big)+O(T^{9/10}\log T).$$

This proves our theorem with the remainder term.

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