37. *Properties of Certain Analytic Functions*

By Hitoshi SAITOH
Department of Mathematics, Gunma College of Technology

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1989)

1. Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$.

Further, we define a function $F_\lambda(z)$ by

$$F_\lambda(z)=(1-\lambda)f(z)+\lambda z f'(z) \tag{1.2}$$

for $\lambda \geqq 0$ and $f(z) \in A(p)$. In the present paper, we derive some properties of functions in the class $A(p)$, and of the function $F_\lambda(z)$ defined by (1.2).

2. Main results. We begin with the statement of the following lemma due to Miller [1].

Lemma. *Let $\phi(u, v)$ be a complex valued function such that*

$$\phi : D \to C, \quad D \subset C \times C \ (C \text{ is the complex plane}),$$

and let $u=u_1+iu_2$, $v=v_1+iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) *$\phi(u, v)$ is continuous in D,*

(ii) *$(1, 0) \in D$ and $\mathrm{Re}\{\phi(1, 0)\}>0$,*

(iii) *for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1+u_2^2)/2$, $\mathrm{Re}\{\phi(iu_2, v_1)\} \leqq 0$.*

Let $p(z)=1+p_1z+p_2z^2+\cdots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\phi(p(z), zp'(z))\}>0 \quad (z \in U),$$

then $\mathrm{Re}\{p(z)\}>0$ $(z \in U)$.

Applying the above lemma, we prove

Theorem 1. *Let a function $f(z)$ defined by* (1.1) *be in the class $A(p)$. If*

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\alpha \quad \left(0 \leqq \alpha < \frac{p!}{(p-j)!};\ z \in U\right),$$

then we have

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}>\frac{1}{(p-j+1)!}\,\frac{(p-j+1)!\,2\alpha+p!}{2(p-j)+3} \quad (z \in U),$$

where $1 \leqq j \leqq p$.

Proof. We define the function $p(z)$ by

$$\frac{(p-j+1)!}{p!}\,\frac{f^{(j-1)}(z)}{z^{p-j+1}}=\beta+(1-\beta)p(z) \tag{2.1}$$

with $\beta=\dfrac{(p-j+1)!\,2\alpha+p!}{p!\{2(p-j)+3\}}$. Then $p(z)=1+p_1z+p_2z^2+\cdots$ is regular in U. Differentiating both sides in (2.1), we obtain

$$(2.2)\qquad \frac{(p-j+1)!}{p!}f^{(j)}(z)=(p-j+1)\beta z^{p-j}+(p-j+1)(1-\beta)z^{p-j}p(z)+(1-\beta)z^{p-j+1}p'(z)$$

and, by using (2.1) and (2.2), we have

$$(2.3)\qquad (p-j+1)!\left\{\frac{f^{(j)}(z)}{z^{p-j}}-\alpha\right\}=p!(p-j+1)\beta-(p-j+1)!\alpha+p!(p-j+1)(1-\beta)p(z)+p!(1-\beta)zp'(z).$$

Hence, in view of $\operatorname{Re}\{f^{(j)}(z)/z^{p-j}\}>\alpha$, we have

$$(2.4)\qquad \operatorname{Re}\{\phi(p(z), zp'(z))\}>0,$$

where $\phi(u,v)$ is defined by

$$(2.5)\qquad \phi(u,v)=p!(p-j+1)\beta-(p-j+1)!\alpha+p!(p-j+1)(1-\beta)u+p!(1-\beta)v$$

with $u=u_1+iu_2$ and $v=v_1+iv_2$. Then we see that

(i) $\phi(u,v)$ is continuous in $D=C\times C$,

(ii) $(1,0)\in D$ and $\operatorname{Re}\{\phi(1,0)\}=(p-j+1)!\{p!/(p-j)!-\alpha\}>0$,

(iii) for all $(iu_2, v_1)\in D$ such that $v_1\leqq-(1+u_2^2)/2$,

$$\operatorname{Re}\{\phi(iu_2,v_1)\}=p!(p-j+1)\beta-(p-j+1)!\alpha+p!(1-\beta)v_1 \leqq p!(p-j+1)\beta-(p-j+1)!\alpha-\frac{p!(1-\beta)(1+u_2^2)}{2}\leqq 0$$

for $\beta=\dfrac{(p-j+1)!2\alpha+p!}{p!\{2(p-j)+3\}}<1$. Consequently, $\phi(u,v)$ satisfies the conditions in lemma. Therefore, we have $\operatorname{Re}\{p(z)\}>0$ $(z\in U)$, that is,

$$\operatorname{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}>\frac{p!}{(p-j+1)!}\beta=\frac{1}{(p-j+1)!}\,\frac{(p-j+1)!2\alpha+p!}{2(p-j)+3}$$

which completes the proof of Theorem 1.

Taking $j=p$ in Theorem 1, we have

Corollary 1. *Let $f(z)\in A(p)$ and suppose*

$$\operatorname{Re}\{f^{(p)}(z)\}>\alpha\quad(0\leqq\alpha<p!;\ z\in U).$$

Then we have

$$\operatorname{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\}>\frac{2\alpha+p!}{3}\quad(z\in U).$$

Letting $j=1$ in Theorem 1, we have

Corollary 2. *Let $f(z)\in A(p)$ and suppose*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>\alpha\quad(0\leqq\alpha<p;\ z\in U).$$

Then we have

$$\operatorname{Re}\left\{\frac{f(z)}{z^{p}}\right\}>\frac{2\alpha+1}{2p+1}\quad(z\in U).$$

Making $p=j=1$ in Theorem 1, we have

Corollary 3. *Let $f(z)\in A(1)$ and suppose*

$$\operatorname{Re}\{f'(z)\}>\alpha\quad(0\leqq\alpha<1;\ z\in U)$$

Then we have

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\}>\frac{2\alpha+1}{3}\quad(z\in U).$$

Corollary 3 is the result by Owa and Obradovic [2].

Next, we prove

Theorem 2. *Let a function $F_\lambda(z)$ defined by (1.2) for $\lambda \geqq 0$ and $f(z) \in A(p)$. If*

$$\operatorname{Re}\left\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad \left(0 \leqq \alpha < \frac{p!(1-\lambda+p\lambda)}{(p-j)!};\ z \in U\right),$$

then

$$\operatorname{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{(p-j)!\,2\alpha + p!\,\lambda}{(p-j)!\,(2-\lambda+2p\lambda)} \quad (z \in U),$$

where $0 \leqq j \leqq p$.

Proof. By the differentiation of $F_\lambda(z)$, we obtain

$$F_\lambda^{(j)}(z) = (1-\lambda+\lambda j) f^{(j)}(z) + \lambda z f^{(j+1)}(z). \tag{2.6}$$

We define the function $p(z)$ by

$$\frac{(p-j)!}{p!}\frac{f^{(j)}(z)}{z^{p-j}} = \beta + (1-\beta)p(z) \tag{2.7}$$

with $\beta = \dfrac{(p-j)!\,2\alpha + p!\,\lambda}{p!(2-\lambda+2p\lambda)}$ $(0 \leqq \beta < 1)$. Then $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in U. Making the differentiation in (2.7), we have

$$\frac{z f^{(j+1)}(z)}{z^{p-j}} - \frac{p!}{(p-j+1)!}\{\beta + (1-\beta)p(z)\} = \frac{p!}{(p-j)!}(1-\beta) z p'(z). \tag{2.8}$$

By using (2.6), (2.7) and (2.8), we obtain

$$\frac{F_\lambda^{(j)}(z)}{z^{p-j}} - \alpha = \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha + \frac{p!(1-\lambda+p\lambda)(1-\beta)}{(p-j)!}p(z) + \frac{p!\,\lambda(1-\beta)}{(p-j)!} z p'(z). \tag{2.9}$$

Hence, in view of $\operatorname{Re}\{F_\lambda^{(j)}(z)/z^{p-j}\} > \alpha$, we have

$$\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0, \tag{2.10}$$

where $\phi(u, v)$ is defined by

$$\phi(u, v) = \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha + \frac{p!(1-\lambda+p\lambda)(1-\beta)}{(p-j)!}u + \frac{p!\,\lambda(1-\beta)}{(p-j)!}v \tag{2.11}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then we see that

(i) $\phi(u, v)$ is continuous in $\boldsymbol{D} = \boldsymbol{C} \times \boldsymbol{C}$,

(ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = \dfrac{p!(1-\lambda+p\lambda)}{(p-j)!} - \alpha > 0$,

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1+u_2^2)/2$

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2, v_1)\} &= \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha + \frac{p!\,\lambda(1-\beta)}{(p-j)!}v_1 \\ &\leqq \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha - \frac{p!\,\lambda(1-\beta)(1+u_2^2)}{2(p-j)!} \leqq 0\end{aligned}$$

for $\beta = \dfrac{(p-j)!\,2\alpha + p!}{p!\,(2-\lambda+2p\lambda)}$. Consequently, $\phi(u, v)$ satisfies the conditions in lemma. Therefore, we have

$\operatorname{Re}\{p(z)\} > 0$ $(z \in U)$, that is,

$$\operatorname{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{p!}{(p-j)!}\beta = \frac{(p-j)!\,2\alpha + p!\,\lambda}{(p-j)!\,(2-\lambda+2p\lambda)}$$

which completes the assertion of Theorem 2.

Taking $j = 0$ in Theorem 2, we have

Corollary 4. *Let a function* $F_\lambda(z)$ *defined by* (1.2) *for* $\lambda \geqq 0$ *and* $f(z) \in A(p)$. *If*

$$\mathrm{Re}\left\{\frac{F_\lambda(z)}{z^p}\right\} > \alpha \quad (0 \leqq \alpha < 1 - \lambda + p\lambda;\ z \in U),$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{2\alpha+\lambda}{2-\lambda+2p\lambda} \quad (z \in U).$$

Putting $j=p$ in Theorem 2, we have

Corollary 5. *Let a function* $F_\lambda(z)$ *defined by* (1.2) *for* $\lambda \geqq 0$ *and* $f(z) \in A(p)$. *If*

$$\mathrm{Re}\{F_\lambda^{(p)}(z)\} > \alpha \quad (0 \leqq \alpha < p!(1-\lambda+p\lambda);\ z \in U),$$

then we have

$$\mathrm{Re}\{f^{(p)}(z)\} > \frac{2\alpha+p!\lambda}{2-\lambda+2p\lambda} \quad (z \in U).$$

Taking $j=1$ in Theorem 2, we have

Corollary 6. *Let a function* $F_\lambda(z)$ *defined by* (1.2) *for* $\lambda \geqq 0$ *and* $f(z) \in A(p)$. *If*

$$\mathrm{Re}\left\{\frac{F'_\lambda(z)}{z^{p-1}}\right\} > \alpha \quad (0 \leqq \alpha < p(1-\lambda+p\lambda);\ z \in U),$$

then we have

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \frac{2\alpha+p\lambda}{2-\lambda+2p\lambda} \quad (z \in U).$$

Making $p=1$ and $j=0$, and $p=1$ and $j=1$ in Theorem 2, we have the following corollaries which were proved by Owa and Nunokawa [3].

Corollary 7. *Let a function* $F_\lambda(z)$ *defined by* (1.2) *for* $\lambda \geqq 0$ *and* $f(z) \in A(1)$. *If*

$$\mathrm{Re}\left\{\frac{F_\lambda(z)}{z}\right\} > \alpha \quad (0 \leqq \alpha < 1;\ z \in U),$$

then we have

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \frac{2\alpha+\lambda}{2+\lambda} \quad (z \in U).$$

Corollary 8. *Let a function* $F_\lambda(z)$ *defined by* (1.2) *for* $\lambda \geqq 0$ *and* $f(z) \in A(1)$. *If*

$$\mathrm{Re}\{F'_\lambda(z)\} > \alpha \quad (0 \leqq \alpha < 1;\ z \in U),$$

then we have

$$\mathrm{Re}\{f'(z)\} > \frac{2\alpha+\lambda}{2+\lambda} \quad (z \in U).$$

References

[1] S. S. Miller: Differential inequalities and Carathéodory functions. Bull. Amer. Math. Soc., **81**, 79–81 (1975).

[2] S. Owa and M. Obradovič: Certain subclasses of Bazilevič functions of type α. Internat. J. Math. & Math. Sci., **9**, 347–359 (1986).

[3] S. Owa and M. Nunokawa: Properties of certain analytic functions. Math. Japon., **33**, 577–582 (1988).