35. An Elementary Proof of an Order Preserving Inequality

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An operator means a bounded linear operator on a Hilbert space. By only using the idea of polar decomposition, here we give an elementary proof of the following "order preserving inequality" in [1].

Theorem. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

$$(1) \qquad (B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$$

holds for each p and q such that $p \geqq 0$, $q \geqq 1$ and $(1+2r)q \geqq p+2r$.

Proof. First of all, we cite (*) by Löwner-Heinz theorem.

$$(*) \qquad A \geqq B \geqq 0 \text{ ensures } A^\alpha \geqq B^\alpha \qquad \text{for any } \alpha \in [0,1].$$

In the case $1 \geqq p \geqq 0$, the result is obvious by (*). We have only to consider $p \geqq 1$ and $q=(p+2r)/(1+2r)$ since (1) for values q larger than $(p+2r)/(1+2r)$ follows by (*). We may assume that A and B are invertible without loss of generality. Let $B^r A^{p/2} = UH$ be the polar decomposition of the invertible operator $B^r A^{p/2}$ where U means the unitary and $H=|B^r A^{p/2}|$. In the case $1 \geqq 2r \geqq 0$, $A^{2r} \geqq B^{2r}$ holds by (*), then for $q=(p+2r)/(1+2r)$

$$\begin{aligned} B^{-r}(B^r A^p B^r)^{1/q} B^{-r} &= B^{-r}(UH^2U^*)^{1/q} B^{-r} = B^{-r} U H^{2/q} U^* B^{-r} \\ &= A^{p/2} H^{-1} H^{2/q} H^{-1} A^{p/2} = A^{p/2} (H^2)^{1/q-1} A^{p/2} \\ &= A^{p/2} (A^{-p/2} B^{-2r} A^{-p/2})^{1-1/q} A^{p/2} \\ &\geqq A^{p/2} (A^{-p/2} A^{-2r} A^{-p/2})^{(p-1)/(p+2r)} A^{p/2} \\ &= A \geqq B, \end{aligned}$$

so we have the following (2) for $q=(p+2r)/(1+2r)$ and for any $r \in [0, 1/2]$

$$(2) \qquad (B^r A^p B^r)^{1/q} \geqq B^{1+2r}.$$

Put $A_1 = (B^r A^p B^r)^{1/q}$ and $B_1 = B^{1+2r}$. Repeating (2) again for $A_1 \geqq B_1 \geqq 0$, $0 \leqq r_1 \leqq 1/2$ and $p_1 \geqq 1$

$$(B_1^{r_1} A_1^{p_1} B_1^{r_1})^{1/q_1} \geqq B_1^{1+2r_1} \qquad \text{for } q_1 = (p_1+2r_1)/(1+2r_1).$$

Put $p_1 = q \geqq 1$ and $r_1 = 1/2$, then

$$(3) \qquad \{B^{2r+1/2} A^p B^{2r+1/2}\}^{1/q_1} \geqq B^{2(1+2r)}.$$

Put $s=2r+1/2$. Then $q_1=(p_1+2r_1)/(1+2r_1)=(p+2s)/(1+2s)$ since $p_1=q$ and $2(1+2r)=1+2s$. Consequently (3) means that (2) holds for $r \in [0, 3/2]$ since $r \in [0, 1/2]$ and $s=2r+1/2$ and repeating this method, (2) holds for each $r \geqq 0$, that is, (1) is shown.

Reference

[1] T. Furuta: $A \geqq B \geqq 0$ assures $(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$ for $r \geqq 0$, $p \geqq 0$, $q \geqq 1$ with $(1+2r)q \geqq p+2r$. Proc. Amer. Math. Soc., **101**, 85–88 (1987).