32. *Some Qualitative Aspects of Transversely Projective Foliations*

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§1. **Introduction.** Let G be a Lie group which acts transitively and real analytically on a q-dimensional real analytic manifold X. A codimension q foliation $\mathcal{F}$ on a manifold M is called a transversely (G, X) foliation if M is covered by a collection of $\mathcal{F}$-distinguished charts with submersions to X for which the transition functions are given by the action of G. They form a special class of foliations which are, for example, rigid in some sense ([3], [9]). They are sources of examples to show the complexity and the diversity of foliations mainly in quantitative points of view. In particular, various secondary characteristic classes are shown not to vanish on suitable transversely (G, X) foliations (see, e.g., [2], [6], [8], [12], [18]).

On the other hand, to understand the qualitative nature of transversely (G, X) foliations is another interesting problem. There are a series of investigations ([1], [7], [11], [13], [16], [17]) to this end for *transversely affine foliations*, i.e. when $(G, X)=(\mathrm{Aff}^+(\boldsymbol{R}), \boldsymbol{R})$, where $\mathrm{Aff}^+(\boldsymbol{R})$ is the group of orientation preserving affine transformations on $\boldsymbol{R}$. One has a considerably good grip on the geometric aspects of transversely affine foliations.

The purpose of the present paper is to provide basic knowledge about the geometric nature of transversely projective foliations. A *transversely projective foliation* is a transversely (G, X) foliation for $(G, X)=(\mathrm{PSL}(2, \boldsymbol{R}), S^1_\infty)$, where $\mathrm{PSL}(2, \boldsymbol{R})=\mathrm{SL}(2, \boldsymbol{R})/\{\pm I\}$ acts on $S^1_\infty=\boldsymbol{R}\cup\{\infty\}$ as linear fractional transformations. Transversely projective foliations constitute a far wider class, and are far more intriguing, than transversely affine foliations. In fact, any transversely oriented C^2 foliation without compact leaves on the unit tangent bundle of a closed oriented surface of genus >1 is transversely projective up to topological conjugacy ([14]). Transversely projective foliations are known to support the Godbillon-Vey class ([8]), while transversely affine foliation do not ([7]).

Given a transversely (G, X) foliations $\mathcal{F}$ on a manifold M, one can associate by means of analytic continuation a developing submersion

$$D : \tilde{M} \to X,$$

where $\tilde{M}$ is the universal covering of M, and a global holonomy homo-

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morphism

$$h : \pi_1(M) \to G$$

such that $D(\gamma x) = h(\gamma) D(x)$ for $x \in \tilde{M}$ and $\gamma \in \pi_1(M)$. The image $\Gamma = h(\pi_1(M))$ is called the global holonomy group of $\mathcal{F}$.

Throughout this paper, the manifold M is to be oriented and closed unless otherwise specified.

In § 2, we give some sufficient conditions for a transversely projective foliation to have a resilient leaf. Also, examples are given to show that the existence of resilient leaves cannot be characterized in terms of the global holonomy groups alone. In § 3, we generalize Levitt's recent result ([13]) on transversely affine foliations about the existence of exceptional leaves.

Details of these results, as well as the discussion of the Markovness ([4], [5], [10], [15]) of exceptional minimal sets of certain transversely projective foliations, will appear elsewhere.

§ 2. **Resilient leaves.** A leaf of a foliation $\mathcal{F}$ is said to be *resilient* if it is nonproper and with contracting holonomy. A foliation $\mathcal{F}$ is said to be *almost without holonomy* if every noncompact leaf of $\mathcal{F}$ has trivial holonomy. Foliations with resilient leaves must necessarily have more or less complicated geometric structures, while foliations almost without holonomy are the simplest of all. In general, there are a variety of foliations which lie between these two classes. The next result is the nonexistence theorem for transversely projective foliations.

Theorem 1. *If a transversely projective foliation $\mathcal{F}$ has no resilient leaves, then $\mathcal{F}$ is almost without holonomy.*

Next let us consider the effect of the existence of resilient leaves on the structure of the global holonomy group Γ. Recall that a transversely affine foliation has no resilient leaf if and only if Γ is (virtually) abelian ([11]). The 'if' part can easily be extended to the case of transversely projective foliations. However the 'only if' part no longer holds. In fact the next theorem shows that the global holonomy group cannot provide enough informations for this end.

Theorem 2. *There are two transversely projective foliations on 3-manifolds with the same global holonomy group, one with resilient leaves and the other without.*

However it is possible to deduce some necessary conditions on the global holonomy group for a transversely projective foliation to have no resilient leaves.

Theorem 3. *If a transversely projective foliation has no resilient leaf, then its global holonomy group is solvable.*

§ 3. **Exceptional leaves.** A leaf L of a foliation $\mathcal{F}$ is said to be *exceptional* if L is neither proper nor locally dense. Let $\mathcal{L}$ be the normal subgroup of $\pi_1(M)$ consisting of the free homotopy classes of loops contained in leaves of $\mathcal{F}$ and with trivial holonomy. The quotient group $\pi_1(M)/\mathcal{L}$ is called the *fundamental group of the leaf space* $M/\mathcal{F}$ and is denoted by $\pi_1(M/\mathcal{F})$. A deep qualitative analysis of Levitt ([13]) on foliations without holonomy gives, as a corollary, a condition on $\pi_1(M/\mathcal{F})$ for a transversely affine foliation $\mathcal{F}$ not to have an exceptional leaf. This has a verbatim generalization to transversely projective foliations as follows.

Theorem 4. *Let $\mathcal{F}$ be a transversely projective foliation on a possibly open, oriented manifold M. If $\pi_1(M/\mathcal{F})$ does not contain a nonabelian free subgroup, then $\mathcal{F}$ cannot have an exceptional leaf.*

Notice that if $\pi_1(M)$ does not contain a nonabelian free subgroup, nor does $\pi_1(M/\mathcal{F})$, hence $\mathcal{F}$ is exceptional leaf free.

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