

30. Continuation of Bicharacteristics for Effectively Hyperbolic Operators^{†)}

By Gen KOMATSU^{*}) and Tatsuo NISHITANI^{**)}

(Communicated by Kôzaku YOSIDA, M. J. A., April 12, 1989)

1. Continuation for two pairs of bicharacteristics. We shall be concerned with regular— C^∞ or analytic—continuation of (null) bicharacteristics of an effectively hyperbolic differential or pseudo-differential operator. Given a hyperbolic principal symbol $p=p(\rho)=p(x, \xi)$ and an effectively hyperbolic characteristic point $\hat{\rho}=(\hat{x}, \hat{\xi})$, we consider bicharacteristics $\rho=\rho(s)$ tending to $\hat{\rho}$ as $s \uparrow +\infty$ or $s \downarrow -\infty$. Our main result is then stated as follows (see also the final paragraph of this section):

Theorem 1. *There are exactly four such bicharacteristics. Two of them are incoming toward the reference point $\hat{\rho}$ with respect to the parameter s , and the other two are outgoing. Furthermore, each one of the incoming (resp. outgoing) bicharacteristics is naturally continued to the other one, and the resulting two curves are C^∞ regular near $\rho=\hat{\rho}$ as submanifolds of the cotangent bundle. These two curves are analytic near $\rho=\hat{\rho}$ whenever $p=p(\rho)$ is assumed to be analytic there.*

It has been known that the Cauchy problem for a hyperbolic operator is C^∞ well-posed for any lower order terms if and only if the principal symbol is effectively hyperbolic at every multiple (necessarily double) characteristic point. Effective hyperbolicity—introduced by Ivrii and Petkov—requires the existence of (necessarily two) non-zero real eigenvalues of the Hamilton map, i.e. the so-called fundamental matrix obtained by linearizing the Hamilton field of the principal symbol. (It turns out that such eigenvalues must be of the form $\pm\lambda$.) It may be natural to ask how this linear algebraic definition is reflected in the dynamical system of bicharacteristics near the reference point. The analysis proving Theorem 1 indicates that such a dynamical system can be regarded as a perturbation of that of the simplest model $p^0(y, \eta)=\eta_0^2-y_0^2$ in $T^*\mathbf{R}^{n+1}$, as far as the incoming and outgoing bicharacteristics are concerned. In this model, we may restrict ourselves to the (y_0, η_0) -plane, in which the origin is a *saddle point* of the Hamilton field of p^0 . Here, $y=(y_0, y')$ and $\eta=(\eta_0, \eta')$.

The C^∞ continuation, as well as the existence, of exactly four such bicharacteristics was observed earlier by Iwasaki [4] using his difficult

^{†)} This work was supported by Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.

^{*}) Department of Mathematics, Osaka University.

^{**)} Department of Mathematics, College of General Education, Osaka University.

analysis [5] on a factorization of the principal symbol. The method employed in [5] is the Nash-Moser implicit function theorem or, rather, its proof. Our method is completely elementary and applies equally to the analytic case as well as the C^∞ case. Details are contained in [6].

2. How to continue the bicharacteristics. We perform a symplectic change of coordinates $(x, \xi) \rightarrow (y, \eta)$ near the point $\rho = \hat{\rho}$. In the new coordinates $\rho = (y, \eta)$ with $\hat{\rho} = (0, 0) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, the principal symbol takes the form

$$p(y, \eta) = \{\eta_0 - \varphi(y, \eta')\}^2 - \psi(y, \eta'), \quad \psi(y, \eta') \geq 0,$$

up to non-vanishing multiplicative factor, where φ and ψ are terms of order ≥ 2 (and hence the quadratic part of $\eta_0^2 - \psi$ corresponds to the Hamilton map); furthermore, $\psi = E + O^3$ with

$$E(y, \eta') = \eta_0^2 + \sum_{j=1}^{n_1} \mu_j (\eta_j^2 + \eta_j'^2) + \mu_0 \sum_{j=n_1+1}^{n_2} \eta_j^2,$$

where $\mu_j, \mu_0 > 0$ and $0 \leq n_1 \leq n_2 \leq n$. We set

$$\pi^\pm(y_0, u) = \pm \Psi(y_0, u)^{1/2} + y_0 \Phi(y_0, u),$$

where $\Phi(y_0, u) = y_0^{-2} \varphi(y_0, y_0 u)$ and $\Psi(y_0, u) = y_0^{-2} \psi(y_0, y_0 u)$; then, π^\pm are smooth (C^∞ or analytic) functions near $(y_0, u) = (0, 0) \in \mathbf{R} \times \mathbf{R}^{2n}$ satisfying

$$p = p^+ p^- \text{ with } p^\pm = \eta_0 - y_0 \pi^\pm(y_0, v/y_0), \quad v = (y', \eta'),$$

a factorization which is valid within an open set of the form $|v|/|y_0| < \text{constant}$. This restriction causes no difficulty, because every incoming or outgoing bicharacteristic satisfies the estimate $|v|/|y_0|^{3/2} < \text{constant}$. (The conclusion of Theorem 1 implies finally the estimate $v = O(y_0^2)$.) Thus, the problem is transposed from the Hamilton field of p to those of p^\pm , where the parameter must be changed from s to $t = t^\pm$. Then, $dy_0/dt = 1$ in both cases, so that we may take y_0 to be the parameter $t = t^\pm$ simultaneously. Therefore, we are led to an initial value problem of the form

$$(IVP) \quad \frac{dv}{dt}(t) = F\left(t, \frac{v(t)}{t}\right), \quad \frac{v(t)}{t} \rightarrow 0 \in \mathbf{R}^{2n} \text{ as } t \rightarrow 0,$$

where $F = F(t, u) = F^\pm$ is a smooth function satisfying $F(0, 0) = 0$; furthermore, the eigenvalues of $(\partial F / \partial u)(0, 0)$ lie on the imaginary axis. Thus, Theorem 1 will be established if we shall show that:

Theorem 2. *Under the conditions on F as above, the two-sided initial value problem (IVP) admits a smooth (C^∞ or analytic) solution $v = v(t)$ near $t = 0$. Furthermore, the uniqueness is valid for each one of the one-sided problems, among solutions of C^1 -class except at the end point $t = 0$.*

Indeed, once $v = v(t)$ is determined, $\eta_0 = \eta_0(t)$ is obtained by indefinite integration; thus achieved is the regular continuation for each pair of the bicharacteristics. It is then easy to see that the tangent lines of the resulting two curves at the reference point $\hat{\rho}$ are generated by eigenvectors of the Hamilton map associated with the non-vanishing real eigenvalues $\pm \lambda$. In the simplest model mentioned in Section 1, these two curves are the straight lines $\eta_0 = \pm y_0$ with $v = 0$, and the changes of parameters

$s \rightarrow t = t^\pm$ satisfy $|t| = e^{-2|s|}$ as $t \rightarrow 0$, equalities which are also perturbed in the general case as follows: given any $\varepsilon > 0$ small, there exist constants $C_\varepsilon^\pm > 0$ such that

$$C_\varepsilon^- \exp\left(\frac{-2}{1-\varepsilon}|s|\right) \leq |t| \leq C_\varepsilon^+ \exp\left(\frac{-2}{1+\varepsilon}|s|\right) \quad \text{as } t \rightarrow 0.$$

3. **The Briot-Bouquet singularity.** In the analytic case, (the existence part of) Theorem 2 is an immediate consequence of a celebrated result of Briot and Bouquet [1]. In fact, setting $u(t) = v(t)/t$, we can write the problem (IVP) as

$$(BB) \quad t \frac{du}{dt}(t) = f(t, u(t)), \quad u(0) = 0 \in \mathbb{R}^N$$

with $f(t, u) = F(t, u) - u$ and $N = 2n$. Then,

The Briot-Bouquet theorem. *If $f = f(t, u)$ is holomorphic near $(t, u) = 0 \in \mathbb{C} \times \mathbb{C}^N$ and satisfies*

$$f(0, 0) = 0, \quad \sigma(M) \cap N = \emptyset,$$

where $M = (\partial f / \partial u)(0, 0)$ and $N = \{1, 2, \dots\}$, then (BB) admits a unique holomorphic solution $u = u(t)$ near $t = 0 \in \mathbb{C}$. Here, $\sigma(M)$ stands for the spectrum of M .

There is also a C^∞ version of the Briot-Bouquet theorem due to de Hoog and Weiss [2], [3] (see also Russell [7]). This is formulated as a one-sided problem

$$(BB)_T \quad t \frac{du}{dt}(t) = f(t, u(t)) \quad \text{for } 0 < t \leq T,$$

where $f, \partial f / \partial u \in C^0([0, T_0] \times B(R_0))$ with $0 < T \leq T_0$ and $B(R_0) = \{v \in \mathbb{C}^N; |v| \leq R_0\}$, $R_0 > 0$. Setting $C_{T,R}^{r,1} = C^r([0, T], B(R)) \cap C^{r+1}(0, T]$ with $0 < R \leq R_0$, we can state:

C^∞ version of the Briot-Bouquet theorem. *Assume that*

$$f(0, 0) = 0, \quad \sigma(M) \subset \{\tilde{\mu} \in \mathbb{C}; \operatorname{Re} \tilde{\mu} < 0\},$$

where $M = (\partial f / \partial u)(0, 0)$. Then, there exist constants $T \in (0, T_0]$ and $R \in (0, R_0]$ such that $(BB)_T$ has a unique solution u in $C_{T,R}^{0,1}$, which satisfies $u(0) = 0$ automatically. Furthermore, $u \in C_{T,R}^{r,1}$ provided $f, \partial f / \partial u \in C^r([0, T] \times B(R))$.

The proof of Theorem 2 will be completed if we shall show that the C^∞ solution for each one of the one-sided problems continues smoothly across $t = 0$ to that for the other one. Namely, we have to show that

$$(\#) \quad u^{(r)}(+0) = u^{(r)}(-0) \quad \text{for } r \in N_0 = \{0, 1, 2, \dots\}.$$

This can be done by using the integral representation for a solution of the linear problem

$$t \frac{du}{dt}(t) - Mu(t) = g(t) \quad \text{for } 0 < t \leq T;$$

the unique solution $u \in C^0[0, T] \cap C^1(0, T]$ is given by

$$u(t) = \mathcal{E}_M g(t) = \int_0^t \left(\frac{t}{s}\right)^M g(s) \frac{ds}{s} = \int_0^1 \left(\frac{1}{s}\right)^M g(ts) \frac{ds}{s}.$$

It is then easy to see that

$$(\mathcal{E}_M g)^{(r)} = \left(\frac{d}{dt} \right)^r \mathcal{E}_M g = \mathcal{E}_{M-rI} g^{(r)},$$

where I stands for the identity matrix; in particular,

$$(\mathcal{E}_M g)^{(r)}(0) = (rI - M)^{-1} g^{(r)}(0).$$

By using this formula, one can examine the validity of (#) and finishes the proof of Theorem 2.

References

- [1] C. Briot et J. C. Bouquet: Recherches sur les propriétés des fonctions définies par des équations différentielles. *J. École Imp. Polytech.* (1), cah. **36** (t. 21), 133–198 (1856).
- [2] F. R. de Hoog and R. Weiss: Difference methods for boundary value problems with a singularity of the first kind. *SIAM J. Numer. Anal.*, **13**, 775–813 (1976).
- [3] —: On the boundary value problem for systems of ordinary differential equations with a singularity of the second kind. *SIAM J. Math. Anal.*, **11**, 41–60 (1980).
- [4] N. Iwasaki: The Cauchy problem for effectively hyperbolic equations (Remarks). *Hyperbolic Equations and Related Topics* (Proc. Taniguchi Intern. Symp., Katata and Kyoto, 1984) (ed. S. Mizohata), Kinokuniya, Tokyo, pp. 89–100 (1986).
- [5] —: The Cauchy problem for effectively hyperbolic equations (a standard type). *Publ. RIMS, Kyoto Univ.*, **20**, 543–584 (1984).
- [6] G. Komatsu and T. Nishitani: Continuation of bicharacteristics for effectively hyperbolic operators (preprint).
- [7] D. L. Russell: Numerical solution of singular initial value problems. *SIAM J. Numer. Anal.*, **7**, 399–417 (1970).