27. On the Existence of the Poles of the Scattering Matrix for Several Convex Bodies

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(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1988)

1. Introduction. Let $\mathcal{O}$ be an open bounded set in $\boldsymbol{R}^3$ with smooth boundary Γ. We set

$$\Omega=\boldsymbol{R}^3-\bar{\mathcal{O}},$$

and suppose that Ω is connected. Consider the following acoustic problem

$$(1.1)\qquad \begin{cases} \Box u(x,t)=\dfrac{\partial^2 u}{\partial t^2}-\Delta u=0 & \text{in } \Omega\times(-\infty,\infty) \\ Bu(x,t)=0 & \text{on } \Gamma\times(-\infty,\infty) \\ u(x,0)=f_1(x) & \\ \dfrac{\partial u}{\partial t}(x,0)=f_2(x) & \end{cases}$$

where $\Delta=\sum_{j=1}^3 \partial^2/\partial x_j^2$. As boundary operator B we shall consider the following two operators,

$$B_D=1 \quad \text{(Dirichlet condition)}$$

and

$$B_N=\sum_{j=1}^3 n_j(x)\partial/\partial x_j \quad \text{(Neumann condition)}$$

where $n(x)=(n_1(x), n_2(x), n_3(x))$ denotes the unit outer normal of Γ at x.

Denote by $\mathcal{S}_\dagger(z)$, $\dagger=D, N$, the scattering matrix for the scatterer $\mathcal{O}$ under the boundary condition $B_\dagger u=0$ (for the definition, see [6]). It is well known that $\mathcal{S}_\dagger(z)$ is an $\mathcal{L}(L^2(S^2))$-valued meromorphic function in the whole complex domain $\boldsymbol{C}$.

As to the modified Lax and Phillips conjecture,[1] that is, *when $\mathcal{O}$ is trapping, there exists $\alpha>0$ such that a slub domain $\{z\,;\operatorname{Im} z<\alpha\}$ contains an infinite number of poles of the scattering matrix*, we have a few examples. Especially for the Dirichlet boundary condition an obstacle consisting of two disjoint convex bodies is the only example ([2, 3]). The purpose of this note is to study the modified Lax and Phillips conjecture in the case that $\mathcal{O}$ consists of several disjoint strictly convex bodies. Our theorem gives a sufficient condition for the existence of such α, which is stated by means of an analytic function defined by using purely geometric informations of Ω.

This work was done during my stay at Massachusetts Institute of Technology. I would like to express my sincere gratitude to Professor Melrose for the invitation and stimulating conversations.

1) The original one is given in [6, page 158], but $\mathcal{O}$ considered in [4] is a counter example.

2. Statement of theorems. Let $\mathcal{O}_j$, $j=1,2,\cdots,j$ be open bounded sets in $\boldsymbol{R}^3$ with smooth boundary Γ_j. We assume the following:

(H.1) Every $\mathcal{O}_j$ is strictly convex, that is, the Gaussian curvature of Γ_j is positive everywhere.

(H.2) For all $\{j_1, j_2, j_3\} \in \{1,2,\cdots,J\}^3$ such that $j_l \neq j_h$ if $l \neq h$, the convex hull of $\bar{\mathcal{O}}_{j_1}$ and $\bar{\mathcal{O}}_{j_2}$ has no intersection with $\bar{\mathcal{O}}_{j_3}$.

We set

$$\mathcal{O}=\bigcup_{j=1}^{J}\mathcal{O}_j. \tag{2.1}$$

Let γ be a periodic ray in Ω. We shall use the following notations:

d_γ: the length of γ,

T_γ: the primitive period of γ,

i_γ: the number of the reflecting points of γ,

P_γ: the Poincaré map of γ.

Concerning the periodic rays in Ω, we have

$$\sharp\{\gamma: \text{periodic ray in } \Omega \text{ such that } d_\gamma < r\} \leq e^{a_0 r}, \tag{2.2}$$

$$|I-P_\gamma| \geq e^{2a_1 d_\gamma}, \tag{2.3}$$

where a_0 and a_1 are positive constants determined by $\mathcal{O}$, and we denote by $|A|$ the determinant of matrix A.

Define functions $F_\dagger(\mu)$, $\dagger=D,N$, by

$$F_\dagger(\mu)=\sum_\gamma(-1)^{a_\dagger i_\gamma}T_\gamma|I-P_\gamma|^{-1/2}e^{-\mu d_\gamma}, \quad a_D=1, \quad a_N=0 \tag{2.4}$$

where the summation is taken over all the periodic rays in Ω. Note that it follows from (2.2) and (2.3) that F_D and F_N are holomorphic in $\{\mu: \mathrm{Re}\,\mu > a_0-a_1\}$.

Theorem 1. *Let $\mathcal{O}$ be an obstacle given by* (2.1) *satisfying* (H.1) *and* (H.2). *If $F_\dagger$, $\dagger=D$ or N, cannot be prolonged analytically to an entire function, there exists $\alpha>0$ such that the scattering matrix $\mathcal{S}_\dagger(z)$ has infinitely many poles in $\{z; \mathrm{Im}\,z<\alpha\}$.*

Theorem 2. *Let $\mathcal{O}_1$, $\mathcal{O}_2$ and $\tilde{\mathcal{O}}_3$ be open sets in $\boldsymbol{R}^3$ satisfying* (H.1) *and* (H.2). *If $\mathcal{O}_3 \subset \tilde{\mathcal{O}}_3$, and the principal curvatures of $\Gamma_3=\partial\mathcal{O}_3$ are greater than κ everywhere of Γ_3, then F_D for $\mathcal{O}=\bigcup_{j=1}^3\mathcal{O}_j$ cannot be prolonged analytically to an entire function. Here κ is a positive constant depending on $\mathcal{O}_1$, $\mathcal{O}_2$ and $\tilde{\mathcal{O}}_3$.*

Remark. It is easy to show that F_N has a singularities on the real axis. Thus in the case of the Neumann condition, the modified Lax and Phillips conjecture holds for $\mathcal{O}$ satisfying (H.1) and (H.2) ([5]).

3. On the proofs of theorems. In order to prove Theorem 1 we shall use the trace formula due to Bardos, Guillot and Ralston [1], and follow the argument in [5]. In the proof of the main estimate of the trace the following lemma is crucial.

Lemma 3. *Let $\rho \in C_0^\infty(-2,2)$ such that $\rho \geq 0$ for all t and $\rho(t)=1$ for $t\in[-1,1]$. Suppose that $F_\dagger$ cannot be prolonged analytically to an entire function. Then there exists a positive constant α_0 such that for any large $\beta>0$ we can find sequences $\{l_q\}_{q=1}^\infty$ and $\{m_q\}_{q=1}^\infty$ with the following properties:*

(i) $l_q \longrightarrow \infty$ *as* $q \longrightarrow \infty$.

(ii) $e^{\beta l_q} \leq m_q \leq e^{2\beta l_q}$.

(iii) $|\langle \rho_q, \hat{F}_\dagger \rangle_{\mathscr{D}(\boldsymbol{R}_+)\times\mathscr{D}'(\boldsymbol{R}_+)}| \geq e^{a_1 l_q}$ *for all* q,

where, $\hat{F}_\dagger$ *is a distribution in* $(0, \infty)$ *given by*

$$\hat{F}_\dagger(t) = \sum_{r} (-1)^{a_\dagger i_r} T_r |I - P_r|^{-1/2} \delta(t - d_r)$$

and $\rho_q(t) = \rho(m_q(t - l_q))$.

In order to show Theorem 2 we shall make a rearrangement of the summation in (2.4), and use the results in [3, 4] on asymptotic behavior of phase functions and periodic rays in Ω.

References

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