25. *A Table of the Dimensions of the Hilbert Modular Type Cusp Forms over Real Quadratic Fields*

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1. Introduction and the statement of result. For a square-free positive number D, let k be a real quadratic number field $\boldsymbol{Q}(\sqrt{D})$, and $\mathfrak{o}$ the ring of the integers in k. Let H^2 be 2-fold product of the complex upper half plane. Let Γ stand for the Hilbert modular group $SL_2(\mathfrak{o})$ embedded in $SL_2(\boldsymbol{R}^2)$. We consider the space $S_2(D)$ of the cusp forms of weight two with respect to Γ.

The purpose of this note is to tabulate the dimensions of $S_2(D)$ for a square-free D and $1<D<1000$. We will publish a large table elsewhere. In the table, the number D is given by

(1) $D=i+j$ ($i=$row number, $j=$column number).

When the mark '-' appears after a figure, $\boldsymbol{Q}(\sqrt{D})$ has a unit of negative norm. The mark '**' means that D is not square-free. To calculate this table, we used ACOS-6 computer system in Okayama University Computer Center.

The trace formula of Hecke operators on $S_2(D)$ was investigated by the author in [3], which was based on more general situations. By [3, Theorem 1], the dimension of $S_2(D)$ is given by

(2) $\dim S_2(D)=t_0+t_1+t_2-1$

where t_0, t_1 and t_2 are the contributions from identity, elliptic elements and parabolic elements, respectively (see below).

Each term can be written as follows.

Theorem.

(3) $t_0=(1/2)\zeta_k(-1)$

(4) $t_1=a(D)h(-D)+b(D)h(-3D)+c(D)$

(5) $t_2=\begin{cases} 0 \text{ if } k \text{ has a unit of negative norm or } D \text{ has no primes } \equiv 3 \bmod 4 \\ -4\Sigma\, h(-d_1)h(-d_2)/w(-d_1)w(-d_2) \quad \text{otherwise}\end{cases}$

where ζ_k is the Dedekind zeta function of k, (d_1, d_2) runs over all discriminants of imaginary quadratic fields satisfying $d_k=d_1\cdot d_2$ (d_k being the discriminant of k), and $h(-d)$, $w(-d)$ denote a class number of $\boldsymbol{Q}(\sqrt{-d})$, an order of the unit group of $\boldsymbol{Q}(\sqrt{-d})$. $a(D)$, $b(D)$ and $c(D)$ are given in the following tables.

D	$D\equiv 1 \bmod (4)$	$D\equiv 2 \bmod (4)$ $D\neq 2$	$D\equiv 3 \bmod (8)$ $D\neq 3$	$D\equiv 7 \bmod (8)$	$D=2$	$D=3$
$8a(D)$	1	3	10	4	5	3

D	$D\equiv 1, 2 \bmod (3)$	$D\equiv 3 \bmod (9)$ $D\neq 3$	$D\equiv 6 \bmod (9)$	$D=3$
$24b(D)$	4	16	8	17

D	$D=5$	$D\neq 5$
$5c(D)$	2	0

2. The method of the computation.

i) t_0. By the functional equation of the Dedekind zeta function, we express t_0 as in (3). Then we can use the formula

(6) $$\zeta_k(-1)=(1/60)\sum \sigma((d_k-b^2)/4)$$

where the summation runs over all b such that $|b|<\sqrt{d_k}$, $b^2\equiv d_k \bmod 4$, and $\sigma(x)$ is the sum of divisors of x.

ii) t_1. There are elliptic points of order 2, 3, 4, 5, 6 in Γ. A point of order 4, 5, or 6 appears only when $D=2, 5$, or 3, respectively. The number of Γ-inequivalent elliptic points of order 2 or 3 is expressed by $h(-D)$ or $h(-3D)$ (cf. [4]). To get t_1, we need calculate a class number of an imaginary quadratic field.

iii) t_2. When k has a unit of negative norm or D has no prime factors which are 3 modulo 4, t_2 vanishes. On the other case, t_2 becomes to

(7) $$-\pi^{-2}d_k^{1/2}\sum_{i=1}^{h_k}\bar{\chi}(A_i^2)L(1,\chi,A_i^2)$$

where χ, $L(s,\chi,A)$ stand for a character of norm signature type, L-series taken over the ideals in A, A_i runs over all ideal classes in $\mathfrak{o}$. It follows from [1, Theorem 1.2] that

(8) $$\sum_i \bar{\chi}(A_i^2)L(1,\chi,A_i^2)=\sum_j L(s,\chi_j)$$

where χ_j runs over all real characters of norm signature type. Therefore t_2 can be expressed as in (3) (cf. [1, § 2]). To calculate t_2 explicitly, we have to get a set of pairs (d_1, d_2) satisfying the conditions in theorem.

Table

	0	100	200	300	400	500	600	700	800	900
1	**	4-	9	12	24-	21	50-	39-	**	59-
2	0-	17	55-	81	130	200	215	**	435	391
3	0	20	41	86	151	166	**	336	339	429
5	0-	1	9	16	**	38	**	45	43	74
6	0	25-	49	**	165	189	264	375	365	471
7	0	21	**	100	126	**	271	271	388	523
9	**	4-	9	12	28-	23-	39	43-	67-	**
10	1-	16	39	101	139	175	304-	277	**	491
11	1	20	66	88	147	231	243	**	449	451
13	0-	4-	5	18-	16	**	35-	40	42	83
14	1	22	64	96-	**	233	253	315	452	470-
15	0	19	46	**	162	179	233	343	356	427
17	0-	**	8	13-	25	22	41-	31	71	44
18	**	24	56-	84	150	169	243	355	358-	**
19	2	17	61	110	144	212	305	307	**	570

Table (continued)

21	0	**	9	18	21-	35-	**	60	46-	85
22	2	24-	48	97	135	**	263	**	377	520-
23	2	23	61	89	**	221	219	325	454	397
26	4-	**	73-	105	147	237	254-	**	461	468
27	**	29	54	94	161	171	257	362	359	**
29	1-	5	9-	14	13	**	31-	**	52-	78-
30	2	32-	53	89	158	196-	**	391-	366	433
31	3	28	53	125	145	**	324	321	414	**
33	0	4	11-	**	29-	25-	45	42-	**	49
34	5	31	**	115	135	227	332-	310	436	553
35	3	**	65	92	135	224	233	**	455	431
37	1-	5-	8	21-	16	34	**	53	**	90-
38	5	23	59	**	153	228-	246	**	427	415
39	4	35	61	113	180	**	**	410	378	531
41	1-	3	13-	12	**	30-	48-	35	**	53-
42	3	29	**	**	181-	184	260	369	385-	458
43	7	24	**	**	151	216	298	302	385	491
45	**	6-	**	13	23-	36	27	66	**	**
46	5	34	73	130-	154	207	333	345-	**	588
47	5	**	74	105	146	246	244	**	**	446
49	**	6-	12	16-	28-	**	53	36	77	65-
51	7	38	69	**	185	196	283	423	407	510
53	2-	**	8	18-	18	36	33-	58	51-	79-
54	**	35	68	120	193	221-	290	408-	403	**
55	6	29	55	117	135	211	318	305	**	533
57	1	6-	12-	9	32-	27-	**	47-	65-	45
58	10-	30	66	127	153-	**	287	324	393	522
59	9	34	85	112	**	258	286	337	505	467
61	2-	4	**	**	21-	39	37-	59-	45	**
62	8	**	75	107-	141	262	268	327	468	444-
63	**	42	68	**	191	202	253	393	385	**
65	2-	3	14-	17-	23	29-	37	**	84-	55-
66	10	43	67	122	208	218	**	408	411	523
67	12	32	73	131	156	**	313	318	**	564
69	1	**	11-	**	24	41-	34	71-	44	87
70	9	41-	**	138-	155	201	308	305	431	593-
71	10	**	87	113	186	283	266	393	498	491
73	2-	7-	9	19-	27	23	57-	38-	**	60
74	15-	38	98-	119	175	261	295	**	495	503
77	2	6	13-	20	**	43-	34-	53	56-	80-
78	8	48	70	**	190	**	279	411-	371	494
79	15	44	**	153	163	245	349	359	460	593
81	**	7-	15-	16	36-	26	54	44	74-	**
82	16-	35	73	132	165	233	313	299	**	576
83	14	42	91	109	151	260	275	**	508	442
85	3-	8-	7	19	23-	**	39-	60-	43	100-
86	16	43	84	135	**	289-	**	396	532	495-
87	12	45	67	**	204	213	300	396	382	479
89	3-	**	**	18-	33	32	52	41	84	58
90	**	44	84-	111	**	218	293	413	421	**
91	17	45	93	141	187	244	360	343	**	639
93	2	9-	12-	22	25-	38-	**	72	46	85

Table (continued)

94	16	51	**	165-	178	**	370	364-	464	601
95	14	35	96	113	**	255	280	363	490	485
97	3-	8-	**	18-	24	28	59-	41-	63	65-
98	**	**	103-	113	186	245	282-	339	515	483
99	**	56	81	135	230	244	325	435	411	**

References

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