15. *On the Erdös-Turán Inequality on Uniform Distribution. II*

By Petko D. PROINOV
Department of Mathematics, University of Plovdiv, Bulgaria

(Communicated by Shokichi IYANAGA, M. J. A., Feb. 12, 1988)

This is continued from [1].

2. To prove Theorem 1 we need three lemmas.

Lemma 1. *Let a function f satisfy the right Lipschitz condition on $\boldsymbol{R}$ with constant L, and let Δ be a closed interval. Set $\delta=\|f\|/2L$, where $\|f\|$ denotes the supremum norm of f on Δ. Then there exists a real number a such that either*

$$f(x+a)\geqq L(x+\delta)\qquad \text{for all } x<\delta \tag{5}$$

or

$$f(x+a)\leqq L(x-\delta)\qquad \text{for all } x>-\delta. \tag{6}$$

Proof. By the assumption, it follows that f is a function of bounded variation on every closed interval. Hence, both limit values $f(x+)$ and $f(x-)$ exist for every $x\in\boldsymbol{R}$. Moreover, we have

$$f(x+)\leqq f(x)\leqq f(x-)\qquad \text{for all } x\in\boldsymbol{R}. \tag{7}$$

Indeed, since f satisfies the right Lipschitz condition with constant L, we have

$$f(x+t)-Lt\leqq f(x)\leqq f(x-t)+Lt$$

for all $x\in\boldsymbol{R}$ and $t>0$. Passing to the limit in these inequalities as $t\to0+$ we obtain (7).

Let us consider f on the closed interval Δ. Then from (7), it follows that there exists a point $b\in\Delta$ such that either $\|f\|=f(b-)$ or $\|f\|=-f(b+)$. Now set

$$a=\begin{cases} b-\delta & \text{if } \|f\|=f(b-),\\ b+\delta & \text{if } \|f\|=-f(b+).\end{cases} \tag{8}$$

We shall prove that the real number a defined by (8) satisfies the requirement of the lemma.

Suppose first that $\|f\|=f(b-)$. Then from the definition of δ, we conclude that $f(b-)=2L\delta$. Now choose two real numbers y and t with $y<t<b$. Since f satisfies the right Lipschitz condition on $\boldsymbol{R}$ with constant L,

$$f(y)\geqq f(t)-L(t-y).$$

Passing to the limit in this inequality as $t\to b-$ we obtain

$$f(y)\geqq f(b-)-L(b-y)=2L\delta-L(b-y). \tag{9}$$

Now let $x<\delta$. Then (8) implies that $x+a<b$. Hence, we can apply (9) with $y=x+a$. Thus, we arrive at

$$f(x+a)\geqq 2L\delta-L(b-a-x)=2L\delta-L(\delta-x)=L(x+\delta),$$

and so, in the considered case, (5) holds.

In the case $\|f\|=-f(b+)$, it can be proved in a similar way that (6) holds. Therefore, in any case, we either have (5) or (6). Q.E.D.

In what follows, for an integrable function f on $[0,1]$ and a positive integer m, we denote by $I_m(f)$ the mth Fejér integral of f, i.e.,

$$I_m(f\,;t)=\int_0^1 f(x)F_m(x-t)dx \qquad \text{for all } t\in \boldsymbol{R}, \tag{10}$$

where

$$F_m(x)=\frac{1}{m+1}\sum_{h=-m}^{m}(m+1-|h|)e^{2\pi i h x} \tag{11}$$

is the mth Fejér kernel. We recall that F_m is a nonnegative even function with $\int_0^{1/2} F_m(x)dx=1/2$.

Lemma 2. *Let a function f be as in Theorem 1. Then there exists a real number a such that the inequality*

$$\|f\|<2L/(m+1)+2|I_m(f\,;a)| \tag{12}$$

holds for any positive integer m.

Proof. We may assume that f satisfies a right Lipschitz condition since the other case follows immediately from this case (replacing f by $-f$). Set $\delta=\|f\|/2L$ ($\|f\|$ is the supremum norm of f on the closed interval $\Delta=[0,1]$). Now extend f on $\boldsymbol{R}$ with period 1. It is easy to prove that the extended function f satisfies the right Lipschitz condition on the whole real line $\boldsymbol{R}$ with constant L. Then according to Lemma 1 there exists a real number a such that either (5) or (6) holds. Further we assume that (5) holds. The other alternative can be treated in a similar way.

Now let m be a given positive integer. We are going to prove (12). We can suppose that $\|f\|\geqq 2L/(m+1)$ since otherwise there is nothing to prove. From the last inequality and the definition of δ, we conclude that $\delta\geqq 1/(m+1)$. Because of the periodicity of f and F_m, we can write the Fejér integral $I_m(f\,;a)$ in the form

$$I_m(f\,;a)=\int_{-1/2}^{1/2} f(x+a)F_m(x)dx. \tag{13}$$

For the value of δ there are two possible cases:

$$1/(m+1)\leqq\delta\leqq 1/2 \quad \text{or} \quad \delta>1/2.$$

Suppose first that $1/(m+1)\leqq\delta\leqq 1/2$. It is known (see [2: Lemma 1]) that in this case,

$$\int_\delta^{1/2} F_m(x)dx<1/6\delta(m+1). \tag{14}$$

From (13), it follows that

$$I_m(f\,;a)=I_1+I_2+I_3, \tag{15}$$

where I_1, I_2 and I_3 denote the integrals of the function $f(x+a)F_m(x)$ on the intervals $[-\delta,\delta]$, $[-1/2,-\delta]$ and $[\delta,1/2]$, respectively. Using (5) and the above mentioned properties of the Fejér kernel we deduce the estimate

$$I_1\geqq L\int_{-\delta}^{\delta}(x+\delta)F_m(x)dx=2L\delta\int_0^{\delta}F_m(x)dx \tag{16}$$

$$=L\delta-2L\delta\int_{\delta}^{1/2}F_m(x)dx$$
$$=\|f\|/2-2L\delta\int_{\delta}^{1/2}F_m(x)dx.$$

Analogously, using the obvious inequality $f(x+a)\geqq-\|f\|$ which holds for all $x\in \boldsymbol{R}$, we deduce

(17) $$I_2\geqq-\|f\|\int_{-1/2}^{-\delta}F_m(x)dx=-2L\delta\int_{\delta}^{1/2}F_m(x)dx$$

and

(18) $$I_3\geqq-\|f\|\int_{\delta}^{1/2}F_m(x)dx=-2L\delta\int_{\delta}^{1/2}F_m(x)dx.$$

From (15), (16), (17) and (18), it follows that

$$I_m(f;a)\geqq\|f\|/2-6L\delta\int_{\delta}^{1/2}F_m(x)dx.$$

Combining this inequality with (14) we get

$$I_m(f;a)>\|f\|/2-L/(m+1),$$

which implies (12).

Now suppose that $\delta>1/2$. Then from (13), (5) and the above mentioned properties of the Fejér kernel, we obtain

$$I_m(f;a)\geqq L\int_{-1/2}^{1/2}(x+\delta)F_m(x)dx$$
$$=2L\delta\int_{0}^{1/2}F_m(x)dx=\|f\|/2,$$

which again implies (12). Q.E.D.

Lemma 3. *Let a function f satisfy the one-sided Lipschitz condition on $[0,1]$ with constant L. Suppose also that*

$$f(0)=f(1) \quad \textit{and} \quad \int_0^1 f(x)dx=0.$$

Then for any positive integer m, we have

(19) $$\|f\|<\frac{2L}{m+1}+\frac{2}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m+1}\right)|\hat{f}(h)|.$$

Proof. Choose a positive integer m. Using (11) and taking into account that $\int_0^1 f(x)dx=0$ we can write the Fejér integral (10) in the form

(20) $$I_m(f;t)=-\frac{1}{2\pi i}\sum_{h=-m}^{m}{}' \frac{m+1-|h|}{(m+1)h}\hat{f}(h)e^{-2\pi i h t},$$

where the prime in the sum indicates that $h=0$ is excluded from the range of summation. From (20), it follows that

(21) $$|I_m(f;t)|\leqq\frac{1}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m+1}\right)|\hat{f}(h)|$$

holds for each $t\in \boldsymbol{R}$. From (21) and Lemma 2, we get (19). Q.E.D.

Proof of Theorem 1. Let f satisfy the assumption of Theorem 1. Then the function f^* defined on $[0,1]$ by

$$f^*(x)=f(x)-\int_0^1 f(t)dt$$

satisfies the assumption of Lemma 3. Applying Lemma 3 to the function f^* and taking into account the relations $[f^*] \leqq 2\|f^*\|$, $[f^*]=[f]$ and $\hat{f}^*=\hat{f}$, we get (4). Q.E.D.

References

[1] P. D. Proinov: On the Erdös-Turán inequality on uniform distribution. I. Proc. Japan Acad., **64A**, 27–28 (1988).

[2] H. Niederreiter and W. Philipp: Berry-Esseen bounds and a theorem of Erdös and Turán on uniform distribution mod 1. Duke Math. J., **40**, 633–649 (1973).